Third Summer School on Dynamical Approaches in Spectral Geometry

Microlocal Methods in Global Analysis

Göttingen, August 27-30, 2018

Titles and Abstracts

Dorothea Bahns & Ingo Witt (University of Göttingen): Introduction to microlocal analysis.

Abstract We will provide a brief introduction to microlocal analysis. Microlocal analysis is the analysis on phase space. It is concerned with the properties of distributions (= generalized functions). Via the Schwartz kernel theorem there is a direct link to linear operators acing between spaces of distributions.

Distributions actually arising in application are often of a special form, as captured by the notion of an oscillatory integral, which in turn allows an in-depth study of the geometric and analytic properties of these distributions.

Applications of microlocal analysis are manifold and range from linear and nonlinear PDEs over spectral and scattering theory to mathematical physics and harmonic analysis, to name just of few.

We plan to cover the following topics: distributions, especially oscillatory integrals, wave front sets, pseudodifferential operators, Fourier integral operators, applications to elliptic and hyperbolic PDEs and in spectral theory.

Further reading

[1] Yu. V. Egorov, *Linear differential equations of principal type*, Contemp. Soviet Math., New York, 1986.

[2] L. Hörmander, *The analysis of linear partial differential operators. I. Distribution theory and Fourier analysis*, Grundlehren Math. Wiss., vol. 256, Springer, Berlin, 1990.

[3] ———, *The analysis of linear partial differential operators. III. Pseudodifferential operators*, Grundlehren Math. Wiss., vol. 274, Springer, Berlin, 1994.

[4] ———, *The analysis of linear partial differential operators. IV. Fourier integral operators*, Grundlehren Math. Wiss., vol. 275, Springer, Berlin, 1994.

[5] R. Melrose, *Geometric scattering theory*, Stanford Lectures, Cambridge Univ. Press, Cambridge, 1995.

[6] M. A. Shubin, *Pseudodifferential operators and spectral theory*, Springer, Berlin, 2001.

[7] C. Sogge, *Hangzhou lectures on eigenfunctions of the Laplacian*, Ann. of Math. Stud., vol. 188, Princeton Univ. Press, Princeton, NJ, 2014.

[8] M. Taylor, *Pseudodifferential operators*, Princeton Math. Ser., vol. 34, Princeton Univ. Press, Princeton, N.J., 1981.

Clara Aldana (University of Luxembourg): Polyakov formulas and heat kernels on surfaces.

Abstract In the first two talks, I will introduce the determinant of the Laplace operator on a closed surface and I will state and prove Polyakov's formula in this context. Then I will mention how microlocal methods appear as a powerful technique when trying to generalize the definition of the determinant and Polyakov's formula to settings that involve singularities of the metric. In the third and four talk, following the theory developed by Richard Melrose, I will define manifolds with corners, blow up spaces, and polyhomogenous conormal distributions. I will explain how to apply these concepts to prove an asymptotic expansion of the heat trace for small times in a particular setting. In the last talk, time permitting, I will explain some parts of the proof of a Polyakov formula for angular sectors that I obtained in joint work with Julie Rowlett.

Further reading

[9] C. Aldana and J. Rowlett, *A Polyakov formula for sectors*, J. Geom. Anal. **28** (2018), 1773-1839.
[10] R. Melrose, *Differential analysis on manifolds with corners*. Manuscript. Available on http://www-math.mit.edu/~rbm/book.html.

Tobias Hartung (King's College London): Zeta functions of Fourier integral operators.

Abstract Traces are an important tool in operator algebras since they give rise to invariants. Depending on the algebra, these invariants may for instance be spectral, geometrical, topological, or physical. Operator ζ-functions are a means of constructing traces by extending a partially defined trace. In this lecture series, we will discuss this trace construction for Fourier integral operators. In particular, we will introduce ζ-functions of gauged poly-log-homogeneous distributions which contain Fourier integral operator ζ-functions as a special case. Furthermore, we will discuss integration theory in a space of ζ-functions since it is often desirable to consider an integrable family of Fourier integral operators A and conclude that the ζ-function commutes with integration, i.e., $\zeta\left(\int A\,d\mu\right) = \int \zeta \circ A\,d\mu$. Finally (and time permitting) we will discuss applications to quantum field theory, where ζ-functions allow for a mathematically rigorous definition of vacuum expectation values from a path integral point of view.

Further reading

[11] V. Guillemin, *Gauged Lagrangian distributions*, Adv. Math. **102** (1993), 184-201.

[12] ———, *Residue traces for certain algebras of Fourier integral operators*, J. Funct. Anal. **115** (1993), 391-417.

[13] ———, *Wave-trace invariants*, Duke Math. J. **83** (1996), 287-352.

[14] T. Hartung, *ζ-functions of Fourier integral operators.* Ph.D. thesis, King's College London, 2015.

[15] ———, *Regularizing Feynman path integrals using the generalized Kontsevich-Vishik trace*, J. Math. Phys. **58** (2017), 123505.

[16] S. W. Hawking, *Zeta function regularization of path integrals in curved spacetime*, Comm. Math. Phys. **55** (1977), 133-148.

[17] M. Kontsevich and S. Vishik, *Determinants of elliptic pseudo-differential operators.* MPIM preprint 1994, arXiv:hep-th/9404046.

[18] ———, *Geometry of determinants of elliptic operators*, Functional Analysis on the Eve of the 21st Century: Volume I (S. Gindikin, J. Lepowsky, and R. Wilson, eds.), Progr. Math., vol. 131, Birkhäuser Boston, Boston, 1995, pp. 173-197.

[19] S. Paycha, *Zeta-regularized traces versus the Wodzicki residue as tools in quantum field theory and infinite dimensional geometry*, Proceedings of the International Conference on Stochastic Analysis and Applications (Hammamet, 2001) (S. Albeverio, A. Boutet de Monvel, and H. Ouerdiane, eds.), Kluwer Acad. Publ., Dortrecht, 2004, pp. 69-84.

[20] M. J. Radzikowski, *Micro-local approach to the Hadamard condition in quantum field theory on curved space-time*, Comm. Math. Phys. **179** (1996), 529-553.

Tobias Weich (University of Paderborn): Dynamical resonances via microlocal analysis.

Abstract Dynamical resonances have been introduced in the 80s by Ruelle and Pollicott in order to explain the convergence of deterministic dynamical systems towards equilibrium.

In this lecture series we will first introduce the notion of dynamical resonances and explain their importance for the assymptotic expansion of correlation functions. We will then explain, how microlocal analysis can be used to define dynamical resonances as a discrete spectrum of a linear operator. This will be first done for a toy model before we pass to more general systems such as Anosov maps and Anosov flows following the approach of Faure, Roy and Sjöstrand [21–23]. At the end of the lecture series we will survey some more recent developements concerning spectral geometry via dynamical resonances [25–27] and if time permits we will also discuss the difficulties appearing in the definition of dynamical resonances on manifolds with hyperbolic cusps [24].

Further reading

[21] F. Faure and N. Roy, *Ruelle-Pollicott resonances for real analytic hyperbolic maps*, Nonlinearity **19** (2006), 1233–1252.

[22] F. Faure, N. Roy, and J. Sjöstrand, *Semi-classical approach for Anosov diffeomorphisms and Ruelle resonances*, Open Math. J. **1** (2008), 35-81.

[23] F. Faure and J. Sjöstrand, *Upper bound on the density of Ruelle resonances for Anosov flows*, Comm. Math. Phys. **308** (2011), 325-364.

[24] Y. Guedes Bonthonneau and T. Weich, *Ruelle resonances for manifolds with hyperbolic cusps*, 2017. Preprint, arXiv:1712.07832.

[25] C. Guillarmou, J. Hilgert, and T. Weich, *Classical and quantum resonances for hyperbolic surfaces*, Math. Ann. **370** (2018), 1231-1275.

[26] ———, *High frequency limits for invariant Ruelle densities*, 2018. Preprint, arXiv:1803.06717.

[27] B. Küster and T. Weich, *Quantum-classical correspondence on associated vector bundles over locally symmetric spaces*, 2017. Preprint, arXiv:1710.04625.

Michał Wrochna (Université Grenoble Alpes): Microlocal methods in quantum field theory on curved spacetimes.

Abstract Quantum Field Theory on curved spacetimes is a physical theory that allows to predict quantum effects induced by spacetime geometry. Its mathematical formulation crucially relies on results in hyperbolic PDEs, in which microlocal analysis plays a central role. The goal of these lectures will be to give an introduction focused on linear fields and their propagators, and explain how various microlocal and global problems are intertwined in this context. I will discuss in particular methods based on pseudo-differential calculus, evolutionary equations and scattering theory, and their applications in the study of quantum fields.

Further reading

[28] S. Hollands and R. M. Wald, *Quantum fields in curved spacetime*, General Relativity and Gravitation: A Centennial Perspective (A. Ashtekar, B. K. Berger, J. Isenberg, and M. MacCallum, eds.), Cambridge Univ. Press, Cambridge, 2015, pp. 513-552.

[29] C. Gérard, *Quantum Field Theory on curved spacetimes.* Unpublished notes. Available on https://www-fourier.ujf-grenoble.fr/˜wrochnam/anr/gerard-lecture_notes_QFT.pdf.

[30] C. Gérard and M. Wrochna, *Hadamard property of the* in *and* out *states for Klein-Gordon fields on asymptotically static spacetimes*, Ann. Henri Poincaré **18** (2017), 2715-2756.

Short Presentations

Tuesday

17:20–17:50	**Yafet Sanchez Sanchez**: Propagators on spacetimes of low regularity.
17:55–18:25	**Amir Vig**: Spectral theory of the ellipse.

Abstracts

Yafet Sanchez Sanchez (MPI Bonn): Propagators on spacetimes of low regularity.

Abstract: In this talk, I will present some motivations and technical difficulties that appear when one tries to do Quantum Field Theory in spacetimes with limited regularity. I will focus particularly on the two-point function of quasi-free states and its relationship with adiabatic states. Because this is work in progress only preliminary results and general strategies will be discussed.

Amir Vig (UC Irvine): Spectral theory of the ellipse.

Abstract: A famous inverse problem posed my Kac in the 1960s is to determine the shape of a drum from the set of resonant frequencies at which it vibrates. In this talk, I will discuss recent results in this direction pertaining to elliptical drumheads. In particular, I will outline a proof of "spectral rigidity" using a new parametrix for the wave propagator, which makes an interesting connection to billiards and Birkhoff s conjecture.

Schedule

	Mon	**Tue**	**Wed**	**Thu**
8:20–9:00	Registration			
9:00–9:50	Bahns	Weich	Aldana	Wrochna
10:00–10:50	Aldana	Witt	Weich	Hartung
10:50–11:20	**Coffee**	**Coffee**	**Coffee**	**Coffee**
11:20–12:10	Wrochna	Hartung	Wrochna	Aldana
12:10–14:00	**Lunch**	**Lunch**	**Lunch**	**Lunch**
14:00–14:50	Bahns	Aldana	Hartung	Wrochna
15:00–15:50	Weich	Q&A	Weich	Q&A
15:50–16:20	**Coffee**	**Coffee**	**Coffee**	**Coffee**
16:20–17:10	Witt	Wrochna	Posters	Weich
17:20–18:10	Hartung	17:20–18:25 Short presentations	Aldana	Hartung
19:00		**BBQ**		