Second summer school within the framework of the

International Doctoral Program in Mathematics at TSU

Dyadic Harmonic Analysis, Martingales, and Paraproducts

Bazaleti, Georgia, September 2-6, 2019

Titles and Abstracts

Ushangi Goginava (Tbilisi State University): Basic harmonic analysis.

Abstract We will provide brief introduction to classical Fourier analysis. The following topics will be covered:

- The conjugate mapping. Integral representation of the conjugate operator.
- The truncated Hilbert transform on L_2.
- The Calderón-Zygmund interval decomposition. The Calderón-Zygmund decomposition.
- The Hardy-Littlewood maximal function.
- The Lebesgue differentiation theorem.
- Existence of the Hilbert transform of integrable functions.
- The Hilbert transform on L_p.
- The Schwartz class and tempered distribution.
- The Fourier transform on L_p.
- Littlewood-Paley theory.

Further reading

[1] A. Torchinsky, *Real-variable methods in harmonic analysis*. Pure Appl. Math., vol. 123. Academic Press, Orlando, FL, 1986.
[2] J. Duoandikoetxea, *Fourier analysis*. Grad. Stud. Math., vol. 29, Amer. Math. Soc., Providence, RI, 2001.

Henri Martikainen (University of Helsinki): Shifts and singular integrals.

Abstract Singular integral operators (SIOs) take the general form

$$Tf(x) = \int_{\mathbb{R}^d} K(x, y) f(y)\, dy,$$

where different assumptions on the kernel K lead to important classes of linear transformations arising across pure and applied analysis. These operators can be classified, among other things, according to the size of the singularity of the kernel K.

In this series of lectures we study selected parts of modern theory of SIOs using the point-of-view of dyadic model operators, with a special emphasis on the so-called dyadic shifts. They are a natural generalisation of the martingale transforms

$$f = \sum_{I \in \mathcal{D}} \langle f, h_I \rangle h_I \mapsto \sum_{I \in \mathcal{D}} \lambda_I \langle f, h_I \rangle h_I, \qquad |\lambda_I| \le 1,$$

and characterise an essential part of the behaviour of SIOs. We will e.g. prove the optimal vector-valued boundedness of SIOs: all L^2 bounded SIOs map $L^p(X) \to L^p(X)$, where $p \in (1, \infty)$ and X is a Banach space satisfying the UMD (unconditional martingale differences) property.

Multi-parameter theory is concerned with kernels whose singularity is spread over the union of all hyperplanes of the form $x_i = y_i$, where $x, y \in \mathbb{R}^d$ are written as $x = (x_i)_{i=1}^t \in \mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_t}$ for a fixed partition $d = d_1 + \ldots + d_t$. On the other hand, classical one-parameter kernels are "singular" (involve "division by zero") exactly when $x = y$. We will also study these multi-parameter SIOs from the point-of-view of dyadic model operators and operator-valued analysis. The operator-valued analysis yields a natural connection to the vector-valued analysis of the one-parameter operators.

Carlos Pérez Moreno (Basque Center for Applied Mathematics, Bilbao): Topics from harmonic analysis related to generalized Poincaré-Sobolev inequalities.

Abstract In these lectures we will develop the concept of the mean oscillation of a function in connection with several important objects in analysis like Poincaré inequalities, BMO and the Hölder-Lipschitz spaces which are intimately connected. We will also show that central results in analysis like Poincaré-Sobolev inequalities and the John-Nirenberg theorem are very much related by means of the concept of self-improving property. The basic tools that we will be using come from harmonic analysis, mainly: the Lebesgue differentiation theorem, the Hardy-Littlewood maximal function and the Calderón-Zygmund

decomposition. Further these ideas will allow us to connect with the A_p theory of weights and we will develop it as much as time allows.

Further reading

[1] C. Pérez, *Calderón-Zygmund theory related to Poincaré-Sobolev inequalities, fractional integrals and singular integral operators.* In: J. Lukes and L. Pick (eds.), Function spaces, nonlinear analysis and applications. Lectures notes of the spring lectures in analysis. Charles University and Academy of Sciences, 1999, pp. 31–94.

[2] L. Grafakos, *Classical Fourier analysis.* 3rd ed., Grad. Texts in Math., vol. 249. Springer, New York, 2014.

[3] J.-L. Journé, *Calderón-Zygmund operators, pseudo-differential operators, and the Cauchy integral of Calderón.* Lecture Notes in Math., vol. 994, Springer, Berlin 1983.

René Schilling (Technical University of Dresden): Martingales and some of their applications in analysis.

Abstract In this series of talks we will introduce martingales and supermartingales in general sigma-finite filtered measure spaces. We discuss their main properties (optional stopping, optional sampling, convergence and regularity/closedness) and use them to show the Radon-Nikodym derivative, the Hardy-Littlewood maximal estimate and further (maximal) estimates for martingales (e.g. Burkholder-Davis and Burkholder-Davis Gundy estimates).

Further reading

[1] R. L. Schilling: *Measures, integrals and martingales.* 2nd ed., Cambridge Univ. Press, Cambridge, 2017. [This text develops the theory of martingales in general sigma-finite spaces, not only in probability spaces.]

[2] R. L. Schilling: *Martingale und Prozesse.* De Gruyter Stud., De Gruyter, Berlin, 2018. [This contains a textbook-style discussion of the Burkholder-Davis-Gundy inequalities. Currently only available in German.]

[3] J. Neveu, *Discrete-parameter martingales.* Revised ed., North-Holland Math. Library, vol. 10. North-Holland, Amsterdam, 1975. [A classic.]

Ingo Witt (University of Göttingen): Probabilistic methods in harmonic analysis.

Abstract In the first two lectures, we will cover topics which are preparatory for the main body of the summer school, among others, basic probabilistic concepts and real interpolation. Then, in the third and fourth lectures, we will exemplarily discuss instances, where these probabilistic concepts are effectively used to establish results in harmonic analysis.

Further reading

[1] R. F. Bass, *Probabilistic techniques in analysis.* Probab. Appl. (N. Y.), Springer, New York, 1995.
[2] S. Bochner, *Harmonic analysis and the theory of probability.* Univ. California Press, Berkeley, 1955.
[3] C. Muscalu and W. Schlag, *Classical and multilinear harmonic analysis. Vol. I.* Cambridge Stud. Adv. Math., vol. 137, Cambridge Univ. Press, Cambridge, 2013.

Pavel Zorin-Kranich (University of Bonn): Paraproducts applied to stochastic integration.

Abstract The main difficulty of stochastic integration is the low regularity of sample paths.

I will show how this difficulty is confronted by the theory of rough paths and reduced to a martingale paraproduct estimate.

Further reading

[1] L. C. Young, *An inequality of the Hölder type, connected with Stieltjes integration.* Acta Math. **67** (1936), 251–282.
[2] P. K. Friz and M. Hairer, *A course on rough paths. With an introduction to regularity structures.* Universitext. Springer, Cham, 2014.
[3] V. Kovač and P. Zorin-Kranich, *Variational estimates for martingale paraproducts.* Preprint 12/2018, arXiv:1812.09763v2.

Short Presentations

Wednesday

16:10–16:30	**Gvantsa Shavardenidze**: On the convergence of Cesàro means of negative order of Vilenkin-Fourier series.
16:35–16:55	**Christian Jäh**: Potential theory for Rockland operators on graded Lie groups.

Abstracts

Gvantsa Shavardenidze (Tbilisi State University): On the convergence of Cesàro means of negative order of Vilenkin-Fourier series.

Abstract: In 1971, Onnewer and Waterman establish sufficient condition which guarantees uniform convergence of Vilenkin-Fourier series of continuous functions. In my talk, we will consider different classes of functions of generalized bounded oscillation and establish sufficient conditions for uniform convergence of Cesàro means of negative order in the terms of these classes of functions.

Christian Jäh (University of Göttingen): Potential theory for Rockland operators on graded Lie groups.

Abstract: We will give a brief introduction to graded Lie groups, emphasizing examples. Then we will discuss the definition of Rockland operators and some of their important properties [1,2]. Finally, we introduce the Riesz- and Bessel-potentials associated to a Rockland operator and, following the seminal work of Folland [2], use them to define and study certain function spaces on graded Lie groups.

Further reading

[1] C. Rockland, *Hypoellipticity on the Heisenberg group – representation-theoretic criteria*. Trans. Amer. Math. Soc. **240** (1978), 1–52.

[2] B. Helffer and J. Nourrigat, *Caracterisation des opérateurs hypoelliptiques homogènes invariants à gauche sur un groupe de Lie nilpotent gradué*. Comm. Partial Differential Equations **4** (1979), 899–958.

[3] G. B. Folland, *Subelliptic estimates and function spaces on nilpotent Lie groups.* Ark. Mat. **13** (1975), 161–207.

Schedule

	Mon	Tue	Wed	Thu	Fri
8:30	Registration				
9:15–10:00	Goginava	Peréz	Martikainen	Schilling	Zorin-Kranich
10:10–10:55	Witt	Goginava	Peréz	Peréz	Schilling
	Coffee	**Coffee**	**Coffee**	**Coffee**	**Coffee**
11:25–12:10	Peréz	Martikainen	Goginava	Martikainen	Witt
	Lunch	**Lunch**	**Lunch**	**Lunch**	**Lunch**
14:00–14:45	Goginava	Witt	Zorin-Kranich	**Excursion**	Peréz
14:55–15:40	Zorin-Kranich	Schilling	Q&A		Q&A
	Coffee	**Coffee**	**Coffee**		**Coffee**
16:10–16:55	Witt	Zorin-Kranich	Short presentations		Schilling
17:05–17:50	Schilling	Martikainen	Martikainen		Zorin-Kranich
18:00	Dinner	Dinner	Dinner	(19:00) Conference dinner	Dinner