C*-Algebras and (K)K-theory in solid-state physics

Preliminaries

completeness

Def A C*-algebra is a Banach algebra $(\mathcal{A}, \|\cdot\|)$ endowed with an antilinear involution $* : \mathcal{A} \to \mathcal{A}$ with the property

$$\|a^* a\| = \|a\|^2 \qquad (*)$$

eg: complex numbers, $M_n(\mathbb{C}) = A$ with the operator norms

$B(\mathcal{H}) :=$ bounded operator on a Hilbert space.
with composition
& op. norm.

Thm (GNS) For any C*-algebra $\mathcal{A}$, $\exists$ an isometric emb.

$$\mathcal{A} \subseteq B(\mathcal{H})$$

Any C*-alg can be represented faithfully as operators on some Hilbert space.

From now on $\mathcal{H}$ will be assumed to be separable

Thm (Gelfand)
Let A be a commutative C*-algebra $1 \in A$, let $\Sigma(A)$ denote the Gelfand spectrum of A (space of characters of A). The Gelfand transform is an isometric isomorphism of A onto $\underset{\substack{\uparrow \\ \text{cont. functions on } \Sigma(A) \\ \| \ \| := \| \ \|_\infty}}{C(\Sigma(A))}$.

NB: Gelfand transform

$$A \to C(\Sigma(A))$$
$$a \mapsto \hat{a} : \omega \mapsto \omega(a) \qquad \text{evaluation functnl.}$$

NB: Equivalence of categories.

let $J \subseteq A$ a closed two-sided ideal in A.

A/J the quotient Banach algebra with involution induced from A is a C^*-algebra.

Thm (isomorphism thm) let A, B C^*-algebras

$\pi: A \to B$ a $*$-homomorphism. then

- $\pi(A)$ is a C^*-subalgebra.
- $\dot{\pi}: A/\ker(\pi) \longrightarrow B$ gives an isometric $*$ isomorphism onto $\pi(A) \subseteq B$

Calkin extension let $\mathcal{H}$ be a sep. Hilbert space

$B(\mathcal{H}) \supseteq K(\mathcal{H})$ C^* algebra inclusion

$Q(\mathcal{H}) := B(\mathcal{H})/K(\mathcal{H})$ is called the Calkin algebra

$$0 \to K(\mathcal{H}) \longrightarrow B(\mathcal{H}) \longrightarrow Q(\mathcal{H}) \to 0$$

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§2. K-theory for C^*-algebras (aka operator K-th)

homology theory in the category of C^*-algebra extending topological K-theory (Atiyah-Hirzebruch)

operator K-theory is a functor K_* associating to every C^*-algebra A two Abelian groups $K_i(A)$ $i=0,1$

functoriality $\varphi: A \to B$ $*$-homom.
$\varphi_*: K_*(A) \longrightarrow K_*(B)$ hom. of Ab. grps

satisfying the following 3 properties

1) homotopy invariant: $\varphi \sim \psi$ homotopy

$$\Rightarrow \varphi_* = \psi_*$$

2) half-exact $\quad 0 \to I \xrightarrow{i} E \xrightarrow{p} A \to 0$

$$K_*(I) \xrightarrow{i_*} K_*(E) \xrightarrow{p_*} K_*(A)$$

is exact in the middle

3) Morita-invariant / stable

let p be a rank 1 proj. in $K(\ell^2(\mathbb{N}))$

$$\begin{aligned} A &\to A \otimes K \\ a &\mapsto a \otimes p \end{aligned}$$

induces an isomorphism in K-theory

$K_0(\mathbb{C}) = \mathbb{Z}$, $K_1(\mathbb{C}) = \{0\}$ $\quad (K^*(pt))$

$K_0(K(\mathcal{H})) = \mathbb{Z}$, $K_1(\mathbb{C}) = \{0\}$

Prop let A be a C^*-algebra, every element in $K_0(A)$ can be represented as $[p]-[q]$

p, q are projections into some $M_r(A^+)$ with $p-q \in M_r(A)$ (for some $r \in \mathbb{Z}$).

if $1 \in A \Rightarrow p, q$ can be chosen in $M_r(A)$

(Wegge-Olsen "Portrait of K-theory")

Higher K-theory $\qquad SA = C_0((0,1), A) = C_0(0,1) \otimes A$

$A \to SA$ is functorial

$K_n(A) := K_0(S^n A)$

$\exists$ maps $\partial_n : K_n(A) \to K_{n-1}(I)$ for any extension

$$0 \to I \to E \to A \to 0$$

Bott-periodicity There are natural isomorphisms

$$K_*(A) \simeq K_*(S^2A)$$

For any ext. $0 \to I \xrightarrow{i} E \xrightarrow{p} A \to 0$ we have a six-term exact sequence

$$\begin{tikzcd}
K_0(I) \arrow[r, "i_*"] & K_0(E) \arrow[r, "p_*"] & K_0(A) \arrow[d, "\partial_{01}"] \\
K_1(A) \arrow[u, "\partial_{10}"] & K_1(E) \arrow[l, "p_*"'] & K_1(I) \arrow[l, "\iota_*"']
\end{tikzcd}$$

example: K-theory groups of spheres

$$K_*(C(S^n)) \simeq K_*(\mathbb{C}) \oplus K_*(S^n\mathbb{C})$$

$$\Rightarrow \quad K_i(S^n\mathbb{C}) = \begin{cases} 0 & i+n = 1 \pmod 2 \\ \mathbb{Z} & i+n = 0 \pmod 2 \end{cases}$$

$K_0(C(S^{2n})) = \mathbb{Z}^2$ $\quad$ $K_1(C(S^{2n})) = \{0\}$

$K_0(C(S^{2n+1})) = \mathbb{Z}$ $\quad$ $K_0(C(S^{2n+1})) = \mathbb{Z}$

Def: the non-trivial generator of $K_*(C(S^n))$ which corresp. to the generator of $K_*(S^n\mathbb{C})$ is called the Bott-element.

§3 Fredholm theory

Let $\mathcal{H}$ be a Hilbert space. A bounded operator $F : \mathcal{H} \to \mathcal{H}$ is fredholm if

1) F has closed range
2) $\ker(F)$ and $\ker(F^*)$ are f. dim. space.

We denote by $\mathrm{Fred}(H)$ the set of fredholm operators.

Recall the Calkin extension:

$$0 \to \mathcal{K}(\mathcal{H}) \to \mathcal{B}(H) \to \mathcal{Q}(H) \to 0$$

Thm (Atkinson)

An operator $F \in B(H)$ is Fredholm iff its image $q(F) \in Q(H)$ is invertible

$F \in B(H)$ fredholm $\iff$ invertible modulo compacts

Def $F \in \mathrm{Fred}(H)$. Define

$$\mathrm{Ind}(F) = \dim(\ker F) - \dim(\ker F^*) \in \mathbb{Z}$$

3.1 Fredholm indices & K-theory

Recall
$$A \to A \otimes \mathcal{K}$$
$$a \mapsto a \otimes p \qquad p \text{ rank } 1$$
$$K_*(A) \to K_*(A \otimes \mathcal{K})$$

p rank one $\quad \mathbb{C} \to \mathcal{K}(\ell^2(\mathbb{N}))$
$\quad z \mapsto zp \quad$ induces iso in K-theory

$$K_0(\mathcal{K}) = \mathbb{Z} \qquad K_1(\mathcal{K}) = \{0\}$$

The map $\mathrm{Tr}_* : K_0(\mathcal{K}(\mathcal{H})) \to \mathbb{Z}$ gives a well-def isomorphism.

(direct limit of $\mathrm{Tr}_*^n : K_0(M_n(\mathbb{C})) \to \mathbb{Z}$) "the trace isomorph"

$$0 \to K(H) \to B(H) \to Q(H) \to 0$$

$\partial_1 : K_1(Q(H)) \to K_0(K(H)) = \mathbb{Z}$ is the index

Def let F be Fredholm. Set

$$[F]_1 := [q(F)(q(F)^* q(F))^{-1/2}] \in K_1(Q(H))$$

we refer to this $[F]_1$ as the K_1 class of the Fred. op F

$$\rightarrow \partial_1([F]_1) = [p \ker F] - [p \ker F^*] \in K_0(K(H))$$

$$tr_*(\partial_1 [F]_1) = \dim(\ker F) - \dim(\ker F^*)$$

$$= \mathrm{Ind}(F)$$

Corollary: $F_0, F_1 \in \mathrm{Fred}(H)$

$\mathrm{Ind}(F_0 F_1) = \mathrm{Ind}(F_0) + \mathrm{Ind}(F_1)$

$\mathrm{Ind}(F_0^*) = -\mathrm{Ind}(F_0)$

invariance under homotopy $F_0 \sim_h F_1 \Rightarrow \mathrm{Ind} F_0 = \mathrm{Ind} F_1$

About the K_1 class of F

lemma let A be a C^*-algebra, $g_0, g_1 \in GL_n(\widehat{A})$ s.t $g_0 \sim_h g_1$ $\Rightarrow$ $g_0(g_0^* g_0)^{-1/2} \underset{h}{\sim} g_1(g_1^* g_1)^{-1/2}$

in $U_n(\widehat{A})$

Polar Decomposition let $T \in B(H)$, there exists a unique partial isometry V such that $T = V(T^*T)^{1/2}$

and $\ker V = \ker T$, $\ker V^* = \ker T^*$

Theorem let $F \in \mathrm{Fred}(H)$. then

$$\mathrm{Ind}(F) = tr_* \circ \partial_1([F]_1).$$

∂_1: boundary map $K_1(Q) \to K_0(K)$

Tr_*: trace hom. $K_0(K) \to \mathbb{Z}$

Pf (sketch)

polar decomp: $F = S(F^*F)^{+1/2}$ for some partial isom.

with $\ker S = \ker F$, $\ker S^* = \ker F^*$.

$$\mathrm{Ind}(F) = \mathrm{Ind}(S)$$

define $q(F) = q(S).(q(F)^* q(F))^{+1/2}$

$$q(S) = q(F)(q(F)^* q(F))^{-1/2} = [F]_1$$

$$\begin{aligned}\partial_1([F]_1) &= [1 - S^*S] - [1 - SS^*] \\ &= [p\ker S] - [p\ker S^*] = \\ &= [p\ker F] - [p\ker F^*]\end{aligned}$$

$$Tr_*(\partial_1(F)_1) = \mathrm{Ind}(F) \qquad \square$$

Proposition (Abstract index thm)

let $0 \to I \xrightarrow{i} E \xrightarrow{p} A \to 0$ be an extension of C^*-algebras.

Assume further that E is represented on some H.S. $\mathcal{H}$

$\pi: E \to B(H)$ st. $I = K(\mathcal{H})$

Then an operator $T \in M_n(E)$ is Fredholm if and only if $p(T)$ is invertible in $M_n(A)$

In this case $\mathrm{Ind}(T) = tr_* \, \partial_1 [p(T)] \in \mathbb{Z}$

class of the inv. element in $M_n(A)$

$$\partial_1 : K_1(A) \to K_0(I) = K_0(\mathcal{K})$$

$$\mathrm{tr}_* : K_0(\mathcal{K}) \to \mathbb{Z} .$$

§4. the Noether – Gohberg – Krein index thm

4.1 the Toeplitz algebra & the Toeplitz ext

$S : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$

$S(\lambda_0, \lambda_1, \lambda_2 \quad) = (0, \lambda_0, \lambda_1, \lambda_2 \ldots)$

& $\{e_n\}_{n \geq 0}$ ONB for $\ell^2(\mathbb{N})$ $\qquad S : e_n \mapsto e_{n+1}$

$S^*(\lambda_0, \lambda_1, \lambda_2, \ldots) = (\lambda_1, \lambda_2, \ldots)$

$\ker S^* = \langle e_0 \rangle$ $\qquad e_0 :=$ vacuum vector

S is an isometry $\qquad S^* S = 1$

$$SS^* = 1 - P_{\ker S^*} =$$

$$= 1 - |e_0\rangle\langle e_0|$$

For $x, y \in \ell^2(\mathbb{N})$ $\qquad |x\rangle\langle y| := \theta_{x,y}$ $\qquad$ rank-one op.

$$\theta_{xy}(z) = x \langle y, z \rangle$$

Fix $k \geq 1$ $\qquad e_k := S^k e_0$

$$p_k := |e_k\rangle\langle e_k| = S^k \cdot p_0 \cdot (S^*)^k$$

Def: The Toeplitz algebra $\mathcal{T} = C^*(S)$ is the smallest C^*-subalg of $B(\ell^2(\mathbb{N}))$ that contains S.

__lemma__ $K(\ell^2(\mathbb{N})) \subseteq \mathcal{T}$ as an ideal.

__Pf__ the matrix units in $B(\ell^2(\mathbb{N}))$ $e_{ij} = |e_i\rangle\langle e_j|$ are in $\mathcal{T}$ because

$$e_{ij} = S^i \overset{\overset{P_0}{\shortparallel}}{|e_0\rangle\langle e_0|} (S^*)^j =$$

$$= S^i (1 - SS^*)(S^*)^j$$

Since $K(\ell^2(\mathbb{N}))$ are the closure of finite rank op. $\square$

__Goal__: show that $C(S^1) \simeq \mathcal{T}/K(\ell^2(\mathbb{N}))$

__Fourier analysis__ Fourier tsf gives $L^2(S^1) \simeq \ell^2(\mathbb{Z})$

$$f \mapsto (\hat{f}_n)_{n\in\mathbb{Z}}$$

The H.S. $L^2(S^1)$ carries a faithful rep of $C(S^1)$ by multiplication operators

$$C(S^1) \ni f \longmapsto M_f : L^2(S^1) \circlearrowleft$$

$$g \mapsto f \cdot g$$

__The Hardy space__

$$H^2(S^1) = \{ f \in L^2(S^1) \mid \hat{f}_n = 0 \quad n<0 \}$$

$$\simeq \ell^2(\mathbb{N})$$

Functions in $L^2(S^1)$ that extend hol to the unit disk

the restriction of the F.T gives a unit iso

$$H^2(S^1) \simeq \ell^2(\mathbb{N}).$$

NB: The operator M_f, $f \in C(S^1)$ __does not__ preserve $H^2(S^1)$

Define $P_+ : L^2(S^1) \to H^2(S^1)$ orth. projection

For $f \in C(S^1)$ define the Toeplitz op with symbol f

$$T_f : P_+ \cdot M_f\big|_{H^2} : H^2(S^1) \to H^2(S^1)$$

Lemma The map $f \mapsto T_f$ defines a continuous linear map from $C(S^1) \to C^*(S) = \mathcal{T}$

Moreover, $T_f = S^* T_f S$

Proof can be checked for $f(z) = z^n$. $\square$

Rmk the map $f \mapsto T_f$ is not a $*$-homom.
Actually there are no $*$-homom $C(S^1) \xrightarrow[\sigma]{} \mathcal{T}$
that split

$$0 \to K \to \mathcal{T} \xrightarrow{\pi} C(S^1) \to 0$$

(The Toeplitz ext. does not split!)

Prop let $f \in C(S^1)$ such that $T_f : H^2 \to H^2$ is invertible. Then $f(z) \neq 0$ $\forall z \in S^1$

$M_f : L^2(S^1) \circlearrowleft$ invertible.

Corollaries . for all $f \in C(S^1)$. $\|T_f\| = \|M_f\|$
$= \|f\|$

- $\inf\{\|T_f + K\| : K \in K(\mathcal{H})\} = \|T_f\| = \|f\|$
- $f, g \in C(S^1)$, $T_f \cdot T_g - T_{fg} \in K(\mathcal{H})$.

Thm There is a short exact sequence of C^*-algebras

$$0 \to K(\ell^2(\mathbb{N})) \to \mathcal{T} \xrightarrow{\pi} C(S^1) \to 0$$

Pf: Need to show $\mathcal{T} / K(\underset{\overset{\shortparallel}{H^2(S^1)}}{\ell^2(\mathbb{N})}) \simeq C(S^1)$

We know that T_z generates $\mathcal{T}$ so the map is surjective.

$$\|[T_f]\| = \inf_{K \in K(\mathcal{H})} \|T_f + K\| = \|f\|$$

so T is isometric → bijective ⇒ isomorphism

let $f \in C(S^1)$, assume f is non vanishing.

$$\frac{f}{|f|} : S^1 \to S^1 \quad \text{well-defined \& continuous.}$$

$\pi_1(S^1) \simeq \mathbb{Z}$ any continuous map $g: S^1 \to S^1$ is homotopic to $h: S^1 \to S^1$ $h(z) = z^k$ for some $k \in \mathbb{Z}$.

Def $f \in C(S^1)$ non vanishing. the winding number of f is the unique integer $w(f)$ for which

$$\frac{f}{|f|} \sim_{\text{homotopy}} u^{w(f)}$$

where $u: z \mapsto z$, $S^1 \to S^1$

Thm: If $f \in C(S^1)$ is non-vanishing ⇒ The Toeplitz operator T_f with symbol f is Fredholm and

$$\underbrace{\operatorname{Ind}(T_f)}_{\text{analytical index}} = -\underbrace{w(f)}_{\text{topological invariant}}.$$

Pf: consequence of abstract index theorem for the sequence

$$0 \to K(H^2) \to \mathcal{T} \to \underline{C(S^1)} \to 0$$

Note: the winding number $w(f)$ can be computed purely topologically: for instance if $f \in C^1(S^1)$

$$\Rightarrow \quad w(f) = \int_{S^1} \frac{f'(z)}{f(z)}\, dz$$

Rmk there is a d-dimensional version of the Toeplitz extension: $d \geq 1$ $(d < \infty)$

$$0 \to K(H^2_d) \to \mathcal{T}_d \xrightarrow{\pi_d} C(S^{2d-1}) \to 0$$

H^2_d is the closure of the space of polynomials in d-commuting variables $\mathbb{C}[z_1 \dots z_d]$ in the norm induced by $\langle z^\alpha, z^\beta \rangle = \frac{\alpha!}{|\alpha|!} \delta_{\alpha\beta}$

with multi-index notation: $z^\alpha = z_1^{\alpha_1} \dots z_d^{\alpha_d}$ for $\alpha \in \mathbb{N}^d$

$\alpha! = \alpha_1! \dots \alpha_d!$

$|\alpha| = \alpha_1 + \dots + \alpha_d$

$M_{z_i} : H^2_d \to H^2_d$ multiplication operators $i = 1 \dots d$

The algebra $\mathcal{T}_d$ is the Arveson – Toeplitz algebra: C*-algebra generated by the d-shift $(M_{z_1} \dots M_{z_d}) = M_z$ on the Drury – Arveson (Hardy) space H^2_d.

(hom. polynomials of degree n in d variables $\longleftrightarrow$ symmetric n-tensors on a d-dim v.space)

$$U : H^2_d \xrightarrow{\cong} \begin{array}{c} \mathcal{F}_{sym}(\mathbb{C}^d) \\ \cap \\ \mathcal{F}(\mathbb{C}^d) \end{array}$$

"compression of the shift"

$$S_i : \mathcal{F}_{sym}(\mathbb{C}^d) \to \mathcal{F}_{sym}(\mathbb{C}^d)$$
$$\xi \mapsto P_{sym}(e_i \otimes \xi)$$

$\{e_i\}$ onb of $\mathbb{C}^d$

$S = (S_1 \dots S_d)$ the commuting d-shift

$$U M_z U^* = S$$

$\mathcal{T}_d :=$ C* subalgebra of $B(\mathcal{F}_{sym}(\mathbb{C}^d))$ gen by the comm. d shift $(S_1 \dots S_d)$

Prop (Lesch - Upmeier)

An operator $T \in M_n(\mathcal{T}_d)$ is fredholm $\Leftrightarrow$ $\pi_d(T)$ is invertible in $M_n(C(S^{2d-1}))$

In that case

$$\mathrm{Ind}(T) := (-1)^d \, b_{2d-1}[\pi_d(T)]$$

$[\pi_d(T)] \in K_1(C(S^{2d-1}))$ class of $\pi_d(T)$

$b_{2d-1} : K_1(C(S^{2d-1})) \to \mathbb{Z}$

Bott isomorphism

Index of a Toeplitz op on H^2_d $\rightsquigarrow$ top invariant of odd spheres.

cf: Prodan – Schulz-Baldes

§5: the <u>Su-Schrieffer-Heeger model</u>

model from solid-state: conducting polymer (polyacetilene)

lattice model with chiral symmetry:

<u>Def</u>: A quantum system described by a Hamiltonian H on $\mathbb{C}^{2N} \otimes \ell^2(\mathbb{Z}^d)$ is said to have chiral symmetric if $\exists$ an involutive unitary $J \otimes 1 = J$

$$JHJ = -H \qquad J^* J = J^2 = 1$$

such a system is said to be in the chiral unitary class (or type AIII)

J has eigenvalues $-1, +1$ assumed to have equal multiplicity

$$J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\Rightarrow \quad H = \begin{pmatrix} 0 & A^* \\ A & 0 \end{pmatrix}$$

A operator on $\mathbb{C}^N \otimes \ell^2(\mathbb{Z}^d)$

$d=1$, <u>SSH model</u>: $\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^n \otimes \ell^2(\mathbb{Z})$

$$H = \frac{1}{2}(\sigma_1 + i\sigma_2) \otimes 1_n \otimes U + \frac{1}{2}(\sigma_1 - i\sigma_2) \otimes 1_n \otimes U^*$$
$$+ \, m\, \sigma_2 \otimes 1_n \otimes \mathbb{1}$$

1_n: identity on $\mathbb{C}^n$ $\qquad$ $\mathbb{1}$ id on $\ell^2(\mathbb{Z})$

$U : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ right shift operator

σ_i : Pauli matrices ($\mathrm{tr}(\sigma_i) = 0$ $\det(\sigma) = -1$)

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$J = \sigma_3 \otimes 1_n \otimes 1 . \qquad JHJ = -H$$

J is the unitary operator implementing a chiral symmetry.

$$\mathcal{F} : \ell^2(\mathbb{Z}) \to L^2(S^1) \oplus$$

$$\mathcal{F} H \mathcal{F}^* = \int_{S^1} \underline{dk\, H_k}$$

$$H_k := \frac{1}{2}(\sigma_1 + i\sigma_2) \otimes 1_n \otimes e^{-ik} + \frac{1}{2}(\sigma_1 - i\sigma_2) \otimes 1_n \otimes e^{ik}$$
$$+ m\sigma_2 \otimes 1_n e$$

in matrix form
$$\begin{pmatrix} 0 & e^{-ik} - im \\ e^{ik} + im & 0 \end{pmatrix} \otimes 1_N$$

eigenvalues of H_k are
$$E_\pm(k) = \pm\sqrt{m^2 + 1 - 2m\sin(k)}$$

N fold degenerate

symmetric around zero (follows from chiral symm. $J_k H_k J_k = -H_k$)

NB: $\mathcal{F}$ diagonalises J_k: $\mathcal{F} J \mathcal{F}^* = \int^{\oplus}_{S_1} dk\, J_k$

$$J_k := \sigma_3 \otimes 1$$

$$\Rightarrow \quad \sigma(H_k) = -\sigma(H_k)$$

There is a central gap around the origin:

$$\Delta = [-E_g, E_g] \qquad E_g = ||m|-1|$$

the Hamiltonian has a spectral gap at zero

$$\chi : \mathbb{R} \to \mathbb{R}$$

functional calculus $\underset{\substack{\uparrow \\ \text{Fermi projection}}}{P_F} = \chi(H \leq 0)$

$$J P_F J = 1 - P_F$$

so we can consider the band Hamiltonian

$$Q = 1 - 2P_F = \mathrm{sign}(H)$$

Q also satisfies $JQJ = -Q$

$Q^2 = 1$ spectrum $\{\pm 1\}$ infinitely degenerate.

chiral sym + $Q^2 = 1$

$$Q = \begin{pmatrix} 0 & U_F^* \\ U_F & 0 \end{pmatrix}$$ for a unitary U_F on $\mathbb{C}^N \otimes \ell^2(\mathbb{Z})$

U_F : Fermi unitary

Rmk: the existence of a fermi unitary is a feature of chiral symmetric gapped Hamiltonians!

$$\mathcal{F} Q \mathcal{F}^* = \int_{S^1}^{\oplus} dk\, Q_k \qquad Q_k = \begin{pmatrix} 0 & \frac{e^{-ik}+im}{|e^{ik}+im|} \\ \frac{e^{+ik}+im}{|e^{ik}+im} & 0 \end{pmatrix} \otimes 1$$

$$Q_k = \begin{pmatrix} 0 & U_k^* \\ U_k & 0 \end{pmatrix}$$

can consider the winding number of the Fermi unitary

$$Ch_1(U_F) = i \int_{S^1} \frac{dk}{2\pi} \operatorname{tr}(U_k^* \partial_k U_k)$$

in our case

$$\underline{Ch_1(U_F)} = \begin{cases} -N & m \in (-1,1) \\ 0 & m \in [-1,1] \end{cases}$$

↗

Bulk invariant of the ground state of the Hamiltonian

invariant under small perturbations of the Hamiltonian.

5.1 Edge states & bulk-boundary corresp.

Introduce an edge to the system by restricting to the Half space $\mathbb{C}^2 \otimes \mathbb{C}^n \otimes \ell^2(\mathbb{N})$

$$\hat{H} = \frac{1}{2}(\sigma_1 + i\sigma_2) \otimes 1_n \otimes S + \frac{1}{2}(\sigma_1 - i\sigma_2) \otimes 1_n \otimes S^*$$

$$+ m\sigma_2 \otimes 1 \otimes 1$$

$S: \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ unilateral right shift

$\hat{J} := \sigma_3 \otimes 1_n \otimes 1_{\ell^2(\mathbb{N})}$ chirality operator

$$\hat{J}\hat{H}\hat{J} = -\hat{H} \Rightarrow \sigma(\hat{H}) = -\sigma(\hat{H})$$

We want to consider invariants of the Half-space Hamiltonian.

spectral gap of H at zero $:= \Delta$

$\delta > 0$ s.t. $[-\delta, \delta] \subseteq \Delta$

look at $\mathcal{E}^\delta :=$ Hilbert space generated by the eigenvectors with eigenvalues $[-\delta, \delta]$

$\mathcal{E}^\delta$ is invariant under $\hat{J}$

$$\mathcal{E}^\delta = \mathcal{E}^\delta_+ \oplus \mathcal{E}^\delta_-$$

boundary invariant by looking at $N_\pm = \dim \mathcal{E}^\delta_\pm$ and considering $N_+ - N_-$

can be computed as a trace

$$\mathrm{tr}(\hat{\mathcal{J}}\,\widehat{P_\delta}) = N_+ - N_-$$

$$P_\delta = \chi(|\hat{H}| \leq \delta)$$

Theorem (bulk-boundary correspondence)

let H be the SSH Hamiltonian on $\mathbb{C}^2 \otimes \mathbb{C}^n \otimes \ell^2(\mathbb{Z})$.

let $\hat{H}$ be its half-space restriction.

If U_F is the Fermi unitary associated to H

with winding number as above $(\mathrm{ch}_1(U_F))$

then

$$\underset{\text{bulk invariant}}{\mathrm{ch}_1(U_F)} = -\underset{\text{edge invariant}}{\underline{\mathrm{Tr}(\tilde{\mathcal{J}}\,\tilde{P}(\delta))}}$$

this equality reflects an equality of K-theory classes for a bulk-algebra & an edge algebra.

Relate this formula to the extension

$$0 \to K(\ell^2(\mathbb{N})) \to \mathcal{T} \to C(S^1) \to 0$$

& the 6-term exact sequence in K-theory.

Recall

$$0 \to K(\ell^2(\mathbb{N})) \to \mathcal{T} \to C(S^1) \to 0$$

Toeplitz extension $\Rightarrow$ Noether - Gohberg - Krein index thm

If $f \in C(S^1)$ non-vanishing $\Rightarrow T_f \in B(H^2)$ is fredholm & $\mathrm{Ind}(T_f) = -\omega(f)$

lecture 3 SSH Hamiltonian (1-dim chiral lattice model)

H, defined $\hat{H}$ to be its half-space restriction.

Thm (bulk - edge corresp.)

$$\mathrm{ch}_1(\underset{\substack{\uparrow \\ \text{Fermi unitary}}}{U_F}) = -\mathrm{tr}(\hat{J}\hat{P}(\delta))$$

Plan: provide a K theoretic interpretation.

+ extension to higher dim.

$$\begin{tikzcd} 0 \arrow[r] & K(\ell^2(\mathbb{N})) \arrow[r] & \mathcal{T} \arrow[r, "ev"] \arrow[d] & C(S^1) \arrow[r] & 0 \\ & & C^*(S) & & \end{tikzcd}$$

$\ell^2(\mathbb{N}) \simeq H^2$

$C^*(S)$ unilateral shift

$C(S^1) = C^*(u \mid uu^* = 1 = u^*u)$

$ev: S \to u$

K-groups of the algebras

$K_0(\mathcal{K}) \cong \mathbb{Z}$ with generator $p_0 = |0\rangle\langle 0|$

$K_0(\mathcal{T}) \simeq \mathbb{Z}$

$K_0(C(S^1)) \simeq \mathbb{Z}$

} with generator the identity

$K_1(\mathcal{K}) = \{0\}$, $K_1(\mathcal{T}) = \{0\}$

$K_1(C(S^1)) \simeq \mathbb{Z}$ generated by fctn with unit winding number

$[p_0] = 1$ in $K_0(\mathcal{K})$

$[p_0] = 0$ in $K_0(\mathcal{T})$

$SS^* = 1 - p_0$

$S^*S = 1$

1 & $1-p$ are Murray-vN equivalent

$\Rightarrow$ same class in K_0

$[p] = [1] - [1-p] = 0$

6 term exact sequence induced by the Toeplitz extension

$$\begin{tikzcd}
K_0(\mathcal{K}) \arrow[r] & K_0(\mathcal{T}) \arrow[r] & K_0(C(S^1)) \arrow[d] \\
K_1(C(S^1)) \arrow[u] & K_1(\mathcal{T}) \arrow[l] \arrow[d, phantom, "\shortparallel"] & K_1(\mathcal{K}) \arrow[l] \arrow[d, phantom, "\shortparallel"] \\
 & \{0\} & \{0\}
\end{tikzcd}$$

$$\begin{tikzcd}
0 \arrow[r] & K_1(C(S^1)) \arrow[r, "\mathrm{Ind}"] \arrow[d, phantom, "\shortparallel"] & K_0(\mathcal{K}) \arrow[r, "i_*"] \arrow[d, phantom, "\cong"] & K_0(\mathcal{T}) \arrow[r, "ev_*"] \arrow[d, phantom, "\cong"] & K_0(C(S^1)) \arrow[r] \arrow[d, phantom, "\shortparallel"] & 0 \\
0 \arrow[r] & \mathbb{Z} \arrow[r] \arrow[d, phantom, "\cong"] & \mathbb{Z} \arrow[r] & \mathbb{Z} \arrow[r] & \mathbb{Z} \arrow[r] \arrow[d, phantom, "\cong"] & 0 \\
 & \ker(i_*) & & & \operatorname{coker}(i_*) &
\end{tikzcd}$$

i_* pushforward $\quad i: \mathcal{K} \to \mathcal{T}$

i_* is the zero map

$i_*[p_0] = 0$

$\Rightarrow$ $\mathrm{Ind}: K_1(C(S^1)) \to K_0(\mathcal{K})$ is an isomorphism

$Tr_*: K_0(\mathcal{K}) \xrightarrow{\sim} \mathbb{Z}$ trace homomorphism.

$$\begin{array}{c} ch_1(U_F) \\ \| \\ [U_F] \in K_1(C(S^1)) \end{array} = \underbrace{[\tilde{\hat{P}}_+(\delta)]_0 - [\hat{P}_-(\delta)]_0}_{\in K_0(\mathcal{K})}$$

the equality follows from the fact that

$$\mathrm{Ind}: K_1(C(S^1)) \to K_0(\mathcal{K}) \text{ is an}$$

isomorphism in the case of the Toeplitz extension.

* * *

6. Crossed products by $\mathbb{Z}$ and Pimsner–Voiculescu exact sequence

let α be an automorphism of a C^*-algebra A.

$\alpha \in \mathrm{Aut}(A)$. $\quad \mathbb{Z} \longrightarrow \mathrm{Aut}(A)$

$\qquad\qquad\quad n \longmapsto \alpha^n$

Def $\quad A \rtimes_\alpha \mathbb{Z}$ is the universal C^*-algebra generated by A and by a unitary u implementing the automorphism; i.e. u satisfies

$$\alpha^n(a) = u^n \cdot a \cdot (u^*)^n$$
$$\forall a \in A, n \in \mathbb{Z}.$$

ex : $\theta \in \mathbb{R}/\mathbb{Q}$, $\tau : \mathbb{Z} \to \mathrm{Aut}(C(S^1))$

$$\tau(n)(f(z)) = f(e^{-2\pi i n\theta} z)$$

$$A_\theta = C^*\left(u, v \;\middle|\; \begin{matrix} uu^* = 1 = u^*u \\ vv^* = 1 = v^*v \end{matrix} \;\middle|\; uv = e^{2\pi i\theta} vu\right)$$

Fact : The crossed product $A \rtimes_\alpha \mathbb{Z}$ is the quotient in the extension

$$\# \quad 0 \to A \otimes K \to \mathcal{T}(A,\alpha) \longrightarrow A \rtimes_\alpha \mathbb{Z} \to 0$$

$\mathcal{T}(A,\alpha)$ is the Pimsner–Voiculescu Toeplitz algebra.

let $\mathcal{T} = C^*(S)$ the Toeplitz algebra. then $\mathcal{T}(A,\alpha)$ is the C^*-subalgebra of $(A \rtimes_\alpha \mathbb{Z}) \otimes \mathcal{T}$ generated by $a \otimes 1$ and $u \otimes S$.

$\#$ induces a K-theory 6 term exact sequence

$$\begin{tikzcd}
K_0(A\otimes K) \arrow[rr] \arrow[dr, phantom, "="] & & K_0(\mathcal{T}(A\alpha)) \arrow[rr] \arrow[d, phantom, "="] & & K_0(A\rtimes\mathbb{Z}) & \\
& K_0(A) \arrow[r, "1-\alpha_*"'] & K_0(A) \arrow[rr, "j_*"'] & & K_0(A\rtimes_\alpha\mathbb{Z}) \arrow[d] \arrow[dr] & \\
& K_1(A\rtimes\mathbb{Z}) \arrow[u] & K_1(A) \arrow[l] & & K_1(A) \arrow[ll, "1-\alpha_*"'] & K_1(A) \arrow[d, phantom, "="] \\
K_1(A\rtimes\mathbb{Z}) \arrow[uuu] & & K_1(\mathcal{T}(A\alpha)) \arrow[ll] & & & K_1(A\otimes K) \arrow[lll]
\end{tikzcd}$$

<u>Prop</u> (PV) $\iota : A \longrightarrow \mathcal{T}(A,\alpha)$ induces an isomorphism at the level of K-theory $K_i(A) \simeq K_i(\mathcal{T}(A,\alpha))$
$i=0,1$

$j : A \hookrightarrow A\rtimes_\alpha\mathbb{Z}$

$$\begin{tikzcd}
K_0(A) \arrow[r, "1-\alpha_*"'] & K_0(A) \arrow[r, "j_*"'] & K_0(A\rtimes_\alpha\mathbb{Z}) \arrow[d] \\
K_1(A\rtimes\mathbb{Z}) \arrow[u] & K_1(A) \arrow[l] & K_1(A) \arrow[l, "1-\alpha_*"']
\end{tikzcd}$$

$A=\mathbb{C}$, $\alpha = id$ $\quad A\rtimes_\alpha\mathbb{Z} \simeq C(S^1)$

$$0 \to K \to \mathcal{T} \to C(S^1) \to 0$$

6.2 PV in solid-state physics.

(cf. Prodan – Schulz-Baldes)

C^*-algebras for half space & bulk observables for d dimensional systems.

$$(*) \quad 0 \longrightarrow E_d \longrightarrow \hat{A}_d \longrightarrow A_d \to 0$$

↙ edge algebra ↓ bulk algebra

the Index map implements the b.e. corr.

(*) is isomorphic to

$$0 \to A_{d-1} \longrightarrow \mathcal{T}(A_{d-1}, \alpha) \to A_{d-1} \rtimes \mathbb{Z} \to 0$$

↗ bulk algebra is ≅ to the bulk algebra for $d-1$ $\rtimes_\alpha \mathbb{Z}$

recall $d=1$

$$0 \to \mathcal{K} \to \mathcal{T} \to C(S^1) \to 0$$

it follows that $K_i(E_d) \simeq K_i(\hat{A}_d) \simeq K_i(A_{d-1})$

↗ bulk algebra in one dim less.

PV exact sequence for computing the K-groups

$$\begin{tikzcd}
K_0(A_{d-1}) \arrow[r, "1-\alpha_*"] & K_0(A_{d-1}) \arrow[r, "i_*"] & K_0(A_{d-1} \rtimes \mathbb{Z}) \arrow[d] \\
K_1(A_{d-1} \rtimes \mathbb{Z}) \arrow[u, "\underset{=}{\mathrm{Ind}}"'] & K_1(A_{d-1}) \arrow[l] & K_1(A_{d-1}) \arrow[l]
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{Ind} : K_1(A_{d-1} \rtimes \mathbb{Z}) \arrow[r] \arrow[d, phantom, "\wr\!\!\mid\!\mid"] & K_0(A_{d-1}) \arrow[d, phantom, "\wr\!\!\mid\!\mid"] \\
K_1(A_d) \arrow[r] & K_0(E_d)
\end{tikzcd}$$

we have constructed a map that relates a K_1-invariant of the bulk algebra (class of a unitary) to a K_0 invariant of the edge algebra (class of a projection).

Rmk when d increases the six term exact sequence becomes less trivial

$d = 2$ $\quad A_{d-1} = C(S^1)$ $\qquad K_0(C(S^1)) = \mathbb{Z}$

$K_1(C(S^1)) = \mathbb{Z}$

Corollary

$$K_j(A_d) \simeq K_j(E_{d+1}) \simeq \mathbb{Z}^{2^{d-1}} \qquad j = 0, 1.$$