Moduli Spaces of Positive Curvature Metrics

DMV Jahrestagung - Sektion Geometrie & Topologie

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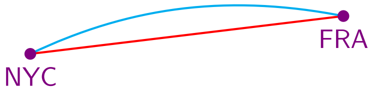
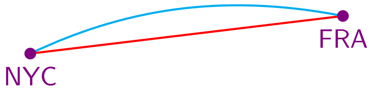
Curvature



Curvature



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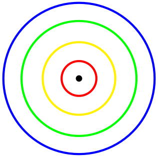

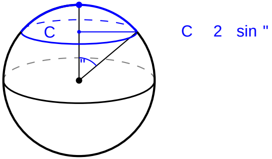

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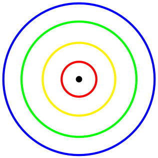
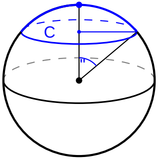
$$\frac{l_P \;\; "}{2 \; "} \quad 1 \quad \frac{\sec_g \;\; P}{6} \, "^2 \quad O \;\; "^3$$ Sectional Curvature

$$\frac{d\,\mathrm{vol}_g}{d\,\mathrm{vol}_{eucl}} \; p \quad "v \quad 1 \quad \frac{\mathrm{Ric} \;\; v; v}{6} \, "^2 \quad O \;\; "^3$$ Ricci Curvature

$$\frac{\mathrm{vol} \;\; B_{"} \;\; p \quad N\, M^d}{\mathrm{vol} \;\; B_{"} \;\; 0 \quad N\, R^d} \quad 1 \quad \frac{\mathrm{scal}_g \;\; p}{6 \;\; d \qquad 2} \, "^2 \quad O \;\; "^3$$ Scalar Curvature





C 2 sin "

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$$\frac{l_P \quad \varepsilon}{2 \quad \varepsilon} \quad 1 \quad \frac{\sec_g \quad P}{6} \varepsilon^2 \quad O \quad \varepsilon^3$$ **Sectional Curvature**

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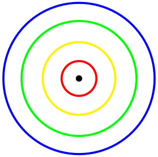
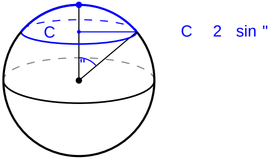
Notions of Curvature





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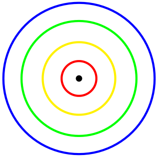

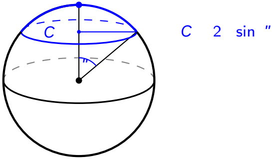

Definition

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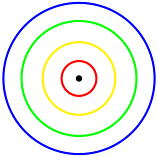

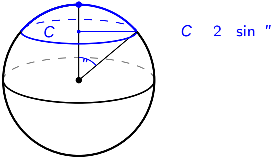

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Examples

Example (Positive Sectional Curvature Metrics)

- S^n L $\mathrm{R}^{n \quad 1}$, $\mathrm{C}P^n; g_{FS}$, $\mathrm{H}P^n; g_{FS}$, $\mathrm{O}P^n$

Example (Positive Ricci Curvature)

- $S^n \quad S^n$ or Σ^n " $bP_{n \quad 1}$ (**Wraith 2011**)

Example (Positive Scalar Curvature)

- $S^2 \quad S^1$

Example (No positive curvature)

- T^n, $K3$-surfaces

Counting Geometries

Question

What is the homotopy type of

$$\mathrm{Riem}^C\, M \qquad \{g \text{ Riem. metric with curv. cond. } C\}?$$

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Example (Hitchin)

Diffeomorphisms act via pull back

$$\mathrm{Di}\quad M \quad " \ \mathrm{Riem}^{C}\ M \qquad \text{via} \qquad ';g \quad (\quad '\ \tilde{}\ g:$$

Hitchin '74: There is $'\ t \quad " \quad {}_{1}\ \mathrm{Di} \quad S^{8}$ such that

$$'\ t\ \ \tilde{}\ g_{\mathrm{round}} \quad \mathrm{j}\ 0\ " \quad {}_{1}\ \mathrm{Riem}^{\mathrm{scal}}\ S^{8}$$

Observer Diffeomorphism

$\mathrm{Di}\ {}_{x_0}\ M$ r ' " Di M D_{x_0} ' idx " $\mathrm{Riem}^C\ M$ freely

Versions of Moduli Spaces

$\mathrm{M}^C\ M$	$\mathrm{Riem}^C\ M$ ©Di M	Moduli Space
$\mathrm{M}^C_{x_0}\ M$	$\mathrm{Riem}^C\ M$ ©Di ${}_{x_0}\ M$	Observer M.S.
$\mathrm{hM}^C\ M$	$\mathrm{Riem}^C\ M$ EDi M ©Di M	homotopy M.S.
$\mathrm{hM}^C_{x_0}\ M$	$\mathrm{Riem}^C\ M$ EDi ${}_{x_0}\ M$ ©Di ${}_{x_0}\ M$	h Obs. M.S.

Properties of Moduli Spaces

$$\begin{array}{ccc}
\mathrm{Riem}^{C}\ M\ ©\mathrm{Di}\ {}_{x_0}\ M & \longrightarrow & \mathrm{Riem}^{C}\ M\ ©\mathrm{Di}\ \ M \\
\uparrow & & \uparrow \\
\mathrm{Riem}^{C}\ M\ ©©\mathrm{Di}\ {}_{x_0}\ M & \longrightarrow & \mathrm{Riem}^{C}\ M\ ©©\mathrm{Di}\ \ M
\end{array} \tag{1}$$

$$\mathrm{Riem}^{C}\ M \longrightarrow \mathrm{hM}\ M \qquad \mathrm{Riem}^{C}\ M\ ©©\mathrm{Di}\ \ M \longrightarrow B\mathrm{Di}\ \ M \tag{2}$$

$$X;\mathrm{hM}\ M \xrightarrow{f\ \ (f\ \tilde{}\ g_{\mathrm{univ}}} \coprod_{E}\ {}_{0}\ \mathrm{Riem}^{C}_{\mathrm{vert}}\ E \tag{3}$$

Theorem (H.)

For all n ' 2 and all odd 3 & j & n 1, we have

$$_2\ \mathrm{h}M^{\mathrm{sec\%0}}_{x_0}\ CP^2\ ;\ g_{FS}\quad \mathrm{j}\ 0; \tag{1}$$

$$_{2j}\ \mathrm{h}M^{\mathrm{sec\%0}}_{x_0}\ CP^n\ ;\ g_{FS}\quad \mathrm{i}\ \mathrm{Q}\ \mathrm{j}\ 0: \tag{2}$$