Asymptotic dimension and some applications to geometric (approximate) group theory

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based on joint work with
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Talk plan

- review $\dim$, introduce asdim for metric spaces/groups,
- introduce approximate groups, see countable approximate groups as metric spaces and define their asdim,
- for the following:

Theorem (Buyalo-Lebedeva, 2007)

For a hyperbolic group G, $\operatorname{asdim} G = \dim \partial G + 1$. In fact, this is true for proper geodesic hyperbolic cobounded metric spaces.

we generalize this to hyperbolic approximate groups:

Theorem (Cordes-Hartnick-T.)

For a hyperbolic approximate group $(\Lambda, \Lambda^\infty)$, $\operatorname{asdim} \Lambda = \dim \partial \Lambda + 1$. In fact, this is true for proper geodesic hyperbolic quasi-cobounded metric spaces.

We will need to introduce some notions: (Gromov) hyperbolicity for metric spaces/groups/approx. groups, (Gromov) boundaries . . .

$\dim$ and asdim, basic comparison

Both dimensions take their values in $\mathbb{N}_0 \cup \{\infty\}$, with $\dim \emptyset := -1$.

$\dim$	asdim
covering (or topological) dimension	asymptotic dimension
H. Lebesgue, 1920's	M. Gromov, 1990's
topological spaces	metric spaces
focused on small stuff	focused on large stuff
open covers	uniformly bounded covers
topological invariant	coarse invariant

Definition of $\dim$

Definition

Let X be a topological space. If $X = \emptyset$, define $\dim X := -1$.
If $X \neq \emptyset$ and $n \in \mathbb{N}_0$, then $\dim X \leq n$ means: for each open cover $\mathcal{U}$ of X there is an open cover $\mathcal{V}$ of X such that

- $\mathcal{V}$ refines $\mathcal{U}$ (i.e., $\forall V \in \mathcal{V}\ \exists U \in \mathcal{U}$ s.t. $V \subseteq U$), and
- $\operatorname{mult} \mathcal{V} \leq n+1$, i.e., any $x \in X$ lies in at most $n+1$ elts. of $\mathcal{V}$.

We say $\dim X := n$ if $\dim X \leq n$ and $\dim X \not\leq n-1$.
If no such n exists, then $\dim X := \infty$.

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We say $\dim X := n$ if $\dim X \leq n$ and $\dim X \not\leq n-1$.
If no such n exists, then $\dim X := \infty$.

Examples:

- $\dim$ (of any discrete space) $= 0$
  In particular, for $\mathbb{Z}^n \subset (\mathbb{R}^n, d_E)$, $\dim \mathbb{Z}^n = 0$.
- $\dim(I^{\aleph_0}) = \infty$, where $I^{\aleph_0} = \prod_{i=1}^{\infty} [0,1]_i$ (Hilbert cube)
  ($I^{\aleph_0}$ with metric $d((x_i),(y_i)) = \sqrt{\sum_{i\in\mathbb{N}} \frac{(d_E(x_i,y_i))^2}{i^2}}$ is bounded)
- $\dim \mathbb{R}^n = n$, $\dim$ (n-manifold) $= n$, $\forall n \in \mathbb{N}$

Definition of asdim

Definition

Let (X, d) be a nonempty metric space and let $n \in \mathbb{N}_0$.
Then $\operatorname{asdim} X \leq n$ means: for each uniformly bounded cover $\mathcal{U}$ of X there is a uniformly bounded cover $\mathcal{V}$ of X such that

- $\mathcal{V}$ coarsens $\mathcal{U}$ (i.e., $\mathcal{U}$ refines $\mathcal{V}$), and
- $\operatorname{mult} \mathcal{V} \leq n + 1$.

We say $\operatorname{asdim} X := n$ if $\operatorname{asdim} X \leq n$ and $\operatorname{asdim} X \not\leq n - 1$.
If no such n exists, then $\operatorname{asdim} X := \infty$.

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If no such n exists, then $\operatorname{asdim} X := \infty$.

Examples:

- asdim (of any bounded metric space) $= 0$
  In particular, for Hilbert cube, $\operatorname{asdim} I^{\aleph_0} = 0$. $[\dim I^{\aleph_0} = \infty]$
- asdim (of a discrete space) can be anything.
  In particular, for $\mathbb{Z}^n \subset (\mathbb{R}^n, d_E)$, $\operatorname{asdim} \mathbb{Z}^n = n$. $[\dim \mathbb{Z}^n = 0]$
- asdim of a discrete group that contains a copy of $\mathbb{Z}^n$, $\forall n \in \mathbb{N}$ is $= \infty$.
- $\operatorname{asdim} \mathbb{R}^n = n$, $\forall n \in \mathbb{N}$ (in fact, $\operatorname{asdim} \mathbb{R}^n = \operatorname{asdim} \mathbb{Z}^n$).

Definition of asdim

Equivalent definition of asdim:

Definition (Coloring definition)

Let (X, d) be a nonempty metric space and let $n \in \mathbb{N}_0$.

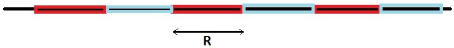
Then $\operatorname{asdim} X \leq n \;\Leftrightarrow\; \forall R > 0$ $(R < \infty)$ there is a uniformly bounded cover $\mathcal{U}$ of X such that

- $\mathcal{U} = \bigcup_{i=1}^{n+1} \mathcal{U}^{(i)}$, where
- each subfamily $\mathcal{U}^{(i)}$ is R-disjoint, i.e., $\forall U \neq U' \in \mathcal{U}^{(i)}$ we have $\operatorname{dist}(U, U') \geq R$.

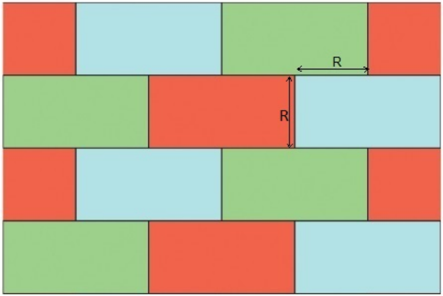
We refer to $i \in \{1, 2, \ldots, n+1\}$ as different colors.

Easy examples of finding asdim

- $\operatorname{asdim} \mathbb{R} = 1$: $\quad \mathcal{U} = \mathcal{U}^{(1)} \cup \mathcal{U}^{(2)}$



- $\operatorname{asdim} \mathbb{R}^2 = 2$: $\quad \mathcal{U} = \mathcal{U}^{(1)} \cup \mathcal{U}^{(2)} \cup \mathcal{U}^{(3)}$



Properties of asdim

Theorem (Monotonicity)

If $A \subseteq X$, then $\operatorname{asdim} A \leq \operatorname{asdim} X$.

Theorem (Product theorem)

$\operatorname{asdim}(X \times Y) \leq \operatorname{asdim} X + \operatorname{asdim} Y$.

Therefore $\operatorname{asdim} \mathbb{R}^n \leq n \cdot \operatorname{asdim} \mathbb{R} = n \cdot 1 = n$.
(Still would have to explain why $\operatorname{asdim} \mathbb{R}^n \not\leq n - 1$.)

Theorem (Functions preserving asdim)

asdim *is a coarse invariant, i.e., it is preserved by coarse equivalences (so, in particular, by quasi-isometries).*

Once we show that $\mathbb{Z}^n \overset{QI}{\approx} \mathbb{R}^n$, they will have the same asdim.

Coarse equivalence and quasi-isometry

Definition

A function $f : (X, d_X) \to (Y, d_Y)$ is a *coarse embedding* if $\exists$ non-decreasing functions $\rho_-, \rho_+ : [0, \infty) \to [0, \infty)$ s.t. $\rho_-(t) \to \infty$ when $t \to \infty$, and $\forall x, x' \in X$ we have

$$\rho_-(d_X(x, x')) \leq d_Y(f(x), f(x')) \leq \rho_+(d_X(x, x')).$$

In particular, if both ρ_- and ρ_+ are linear, i.e., $\exists K \geq 1,\ C \geq 0$ s.t.

$$\frac{1}{K} \cdot d_X(x, x') - C \leq d_Y(f(x), f(x')) \leq K \cdot d_X(x, x') + C,$$

we say that f is a *quasi-isometric embedding* (QI-embedding, or, more precisely, a (K, C)-QI-embedding).

(For $K = 1, C = 0$: f is an isometric embedding.)

Coarse equivalence and quasi-isometry

Definition

If $\exists D \geq 0$ is such that $Y = N_D(f(X))$, i.e., $y \in Y$ is at most D-distant from some element of $f(X)$, we say that f is *coarsely surjective* (and that $f(X)$ is quasi-dense or coarsely dense in Y).

Definition

- If $f : X \to Y$ is a QI-embedding and f is coarsely surjective, then f is called a *quasi-isometry (shortly QI)*. $((K, C, D)$-QI$)$
- If $f : X \to Y$ is a coarse embedding and f is coarsely surjective, then f is called a *coarse equivalence (shortly CE)*.

Properties of metric spaces which are preserved by quasi-isometries are called *QI-invariants*, and properties preserved by coarse equivalences are called *coarse invariants*.

If there exists a quasi-isometry (coarse equivalence) between spaces X and Y, we write $X \overset{QI}{\approx} Y$ ($X \overset{CE}{\approx} Y$).

Coarse equivalence and quasi-isometry

Example: $\mathbb{Z} \hookrightarrow \mathbb{R}$ is a QI with constants $K = 1, C = 0, D = 1$.

Therefore $\mathbb{Z} \overset{QI}{\approx} \mathbb{R}$. Recall the theorem

Theorem (Functions preserving asdim)

asdim *is a coarse invariant, i.e., it is preserved by coarse equivalences (in particular, by quasi-isometries).*
That is, $X \overset{CE}{\approx} Y \Rightarrow \operatorname{asdim} X = \operatorname{asdim} Y$
(in particular, $X \overset{QI}{\approx} Y \Rightarrow \operatorname{asdim} X = \operatorname{asdim} Y$*) .*

Consequently $\operatorname{asdim} \mathbb{Z} = \operatorname{asdim} \mathbb{R}$.

Note: a CE between geodesic metric spaces is a QI.

Metric on groups: finitely generated groups

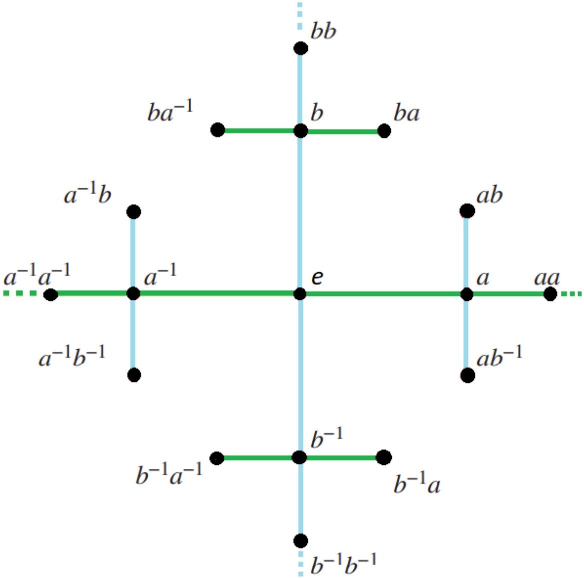
To introduce asdim on groups, we need a metric.

Let G finitely generated group, S a fin.gen. set of G ($S^{-1} = S$).

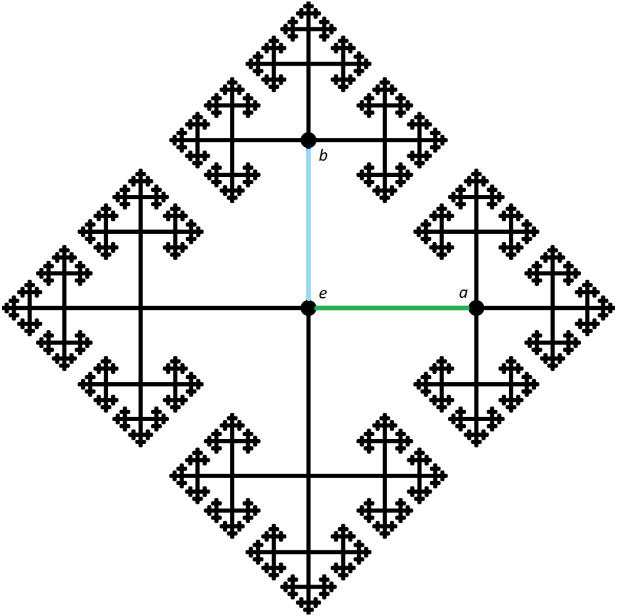
- 1^{st} way, on G we introduce the word metric associated to S: $d_S(g, h) := ||g^{-1}h||_S$ (length of $g^{-1}h$ w.r. to S), $\forall g, h \in G$.
    - d_S is left-invariant: $d_S(ag, ah) = d_S(g, h)$, $\forall a, g, h \in G$,
    - (G, d_S) is a discrete metric space,
    - (G, d_S) is proper (closed balls are compact).
- 2^{nd} way, build the Cayley graph $\Gamma_S(G)$:
    - Vertices: elements of G,
    - Edges: $(g, h) \in E$ if $h = gs$, $s \in S$,
    - metric on $\Gamma_S(G)$: path-length metric, i.e., $d(a, b)$= length of shortest path between a, b. (Each edge of length 1.)
    - $(\Gamma_S(G), d)$ is a geodesic metric space.

  Turns out: d on $V(\Gamma_S(G))$ and d_S on G coincide, and G (identified with $V(\Gamma_S(G))$) is QI to $\Gamma_S(G)$, for any finite generating set S.

Cayley graph for $\Gamma_{\{a,b,a^{-1},b^{-1}\}}(F_2)$



Cayley graph for $\Gamma_{\{a,b,a^{-1},b^{-1}\}}(F_2)$, but fancier



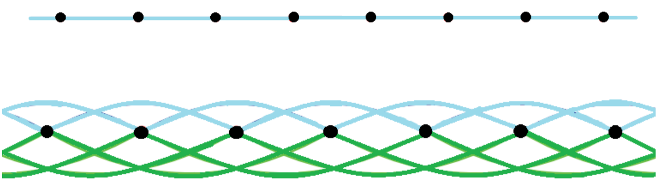
Cayley graph depends on choice of the (fin.) generating set S, but:

Theorem

If S and S' are both finite generating sets for G, then

$$(G, d_S) \overset{QI}{\approx} (\Gamma_S(G), d_S) \overset{QI}{\approx} (\Gamma_{S'}(G), d_{S'}) \overset{QI}{\approx} (G, d_{S'}).$$

Example: $\Gamma_{\{1,-1\}}(\mathbb{Z})$ and $\Gamma_{\{2,3,-2,-3\}}(\mathbb{Z})$.



asdim of finitely generated groups

Definition

For a finitely generated group G, and any fin.gen. set S of G:

$$\operatorname{asdim} G := \operatorname{asdim}(G, d_S) = \operatorname{asdim}(\Gamma_S(G), d_S)$$

Note: asdim is a coarse invariant (in particular, preserved by QI), so definition does not depend on the choice of fin.generating set S.

We can also define $\operatorname{asdim} G := \operatorname{asdim}([G]_c)$, where $[G]_c = \{(X, d_X) \mid (X, d_X) \overset{CE}{\approx} (G, d_S)\}$.

What if G is not finitely generated? Then it can be:

- G countable (not fin.gen.), or
- G uncountable (we will not be covering these today)

(A finitely generated group can have a subgroup which is not finitely generated (but it will be countable).)

For G countable: can define a left-invariant proper metric d:

Definition

Let G be a countable group and $S \subseteq G$ be a symmetric subset. A function $w : S \cup \{e\} \to [0, \infty)$ is called a *weight function on* S if it is proper and satisfies $w^{-1}(0) = \{e\}$ and $w(s) = w(s^{-1})$ for all $s \in S$.

Lemma

Let S be a symmetric generating set of a countable group G and let $w : S \cup \{e\} \to [0, \infty)$ be a weight function. Then

$$\|g\|_{S,w} := \inf \left\{ \sum_{i=1}^{n} w(s_i) \;\middle|\; g = s_1 \cdots s_n,\ s_i \in S \right\}$$

defines a norm on G, and the associated metric $d_{S,w}$ given by $d_{S,w}(g, h) := \|g^{-1}h\|_{S,w}$ is left-invariant and proper.

asdim of countable groups

Theorem

If d_1 and d_2 are two left-invariant proper metrics on a countable group G, then the identity $id : (G, d_1) \to (G, d_2)$ is a coarse equivalence (so $(G, d_1) \overset{CE}{\approx} (G, d_2)$).

So the following makes sense:

Definition

The coarse class $[G]_c$ of a countable group G is the coarse equivalence class of the metric space (G, d), where d is some (hence any) left-invariant proper metric on G.

- Therefore, for a countable group G, define:

$\operatorname{asdim} G := \operatorname{asdim}(G, d)$, where d is any left-invariant proper metric on G. We can also define $\operatorname{asdim}([G]_c) := \operatorname{asdim}(G, d)$, so $\operatorname{asdim} G = \operatorname{asdim}(G, d) = \operatorname{asdim}([G]_c)$.

Definition (Approximate subgroup, T. Tao, 2008)

Let $(G, \cdot)$ be a group and let $k \in \mathbb{N}$. A subset Λ of G is called a *k-approximate subgroup* of G if:

(AG1) $\Lambda = \Lambda^{-1}$ and $e \in \Lambda$, and

(AG2) $\exists$ a finite subset $F \subseteq G$ s.t. $\Lambda^2 \subseteq \Lambda F$ and $|F| = k$.

We say Λ is an *approximate subgroup* if it is a k-approximate subgroup, for some $k \in \mathbb{N}$.

Note:

- $\Lambda^2 = \Lambda \cdot \Lambda = \{a \cdot b \mid a, b \in \Lambda\}$, $\Lambda \cdot F = \{a \cdot f \mid a \in \Lambda, f \in F\}$.
- If Λ is an approx. subgroup, then $\Lambda^\infty := \cup_{k \in \mathbb{N}} \Lambda^k$ is a group ($\Lambda^\infty \leq G$). We call Λ^∞ *the enveloping group of* Λ.

We call the pair $(\Lambda, \Lambda^\infty)$ an *approximate group*.
We say: $(\Lambda, \Lambda^\infty)$ is finite (countable) if Λ is finite (countable).

Approximate groups – examples

(1) Let $(G, \cdot) = (\mathbb{Z}, +)$, $n \in \mathbb{N}$ and define $\Lambda := \{-n, -n+1, \ldots, -1, 0, 1, \ldots, n-1, n\}$.
Then $\Lambda + \Lambda = \{-2n, \ldots, 2n\} \not\subseteq \Lambda$, but for $F = \{-n, n\}$ we get $\Lambda + \Lambda = \Lambda + F$, i.e., Λ is a 2-approximate subgroup of $\mathbb{Z}$.
Also: $\Lambda^\infty = \mathbb{Z}$. Therefore $(\Lambda, \mathbb{Z})$ is an approximate group.

(2) **(Non-example)**: Let $(G, \cdot) = (\mathbb{Z}, +)$ and define $\Lambda := \{2^i \mid i \in \mathbb{Z}\} \cup \{0\} \cup \{-2^i \mid i \in \mathbb{Z}\}$. Then $\Lambda + \Lambda$ contains $2^n + 2^{n+1} = 3 \cdot 2^n$, $\forall n \in \mathbb{N}$, so it contains infinitely many numbers which are not in Λ, and the "distance" of these new numbers to Λ goes to ∞. If F is a *finite* set $\subseteq \mathbb{Z}$, then the "distance" between the numbers in $\Lambda + F$ to Λ is bounded. Therefore we cannot have $\Lambda + \Lambda \subseteq \Lambda + F$, i.e., Λ is not an approximate subgroup of $\mathbb{Z}$.

Approximate groups – examples

(3) If G is a group and $H \leq G \Rightarrow$, then H is also an approximate subgroup of $G \Rightarrow$ the pair (H, H) is an approximate group.

(4) If G is a group and F is a finite symmetric subset of G which contains $e \Rightarrow (F, F^\infty)$ is an approximate group.

(5) If Λ is an approximate subgroup of a group G, then Λ^k is also an approximate subgroup of G, so $(\Lambda^k, \Lambda^\infty)$ is an approximate group.

(6) Cartesian product of two approximate subgroups is an approximate subgroup, the image of an approximate subgroup (via a group homomorphism) is an approximate subgroup.

Approximate groups – examples

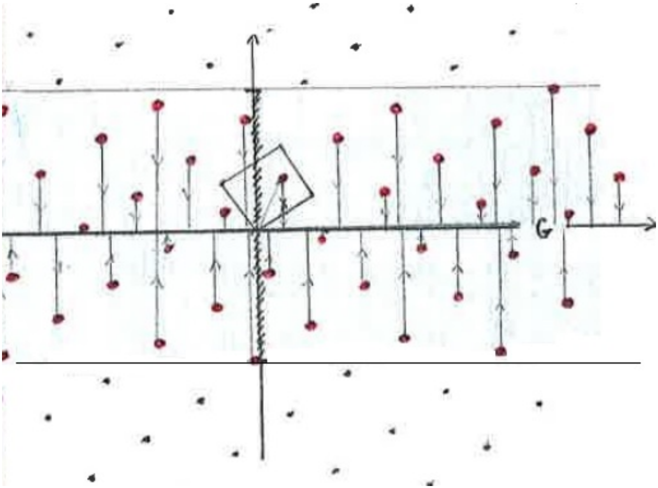
(7) Let $\mathrm{BS}(1,2) = \langle a, b \mid bab^{-1} = a^2 \rangle$ be the Baumslag-Solitar group of type $(1,2)$, and define $\Lambda := \langle a \rangle \cup \{b, b^{-1}\}$. Then Λ is symmetric, contains e and generates $\mathrm{BS}(1,2)$ (so $\Lambda^\infty = \mathrm{BS}(1,2)$). A calculation (using $(b^{-1}ab)^2 = a$) shows that

$$\Lambda^2 \subseteq \Lambda\{e, b, b^{-1}, b^{-1}a\},$$

hence $(\Lambda, \Lambda^\infty)$ is an approximate group.

(8) If G is a locally compact group and W is a relatively compact (i.e. having compact closure) symmetric neighborhood of identity e in G, then (W, W^∞) is an approximate group.

(9) "Cut and project" construction on an irrational lattice in $\mathbb{R}^2$:



Countable approximate groups and their asdim

For a countable approx. group $(\Lambda, \Lambda^\infty)$, how do we define $\operatorname{asdim} \Lambda$?

Recall: for a countable group G:

- there are left-invariant proper metrics on G, and
- if d_1 and d_2 are two left-invariant proper metrics on G, then $(G, d_1) \overset{CE}{\approx} (G, d_2)$,
- asdim is a coarse invariant, so
- $\operatorname{asdim} G := \operatorname{asdim}(G, d)$ $(= \operatorname{asdim}([G]_c))$ is well-defined (for any left-invariant proper metric d on G)

Analogously, if $(\Lambda, \Lambda^\infty)$ is a countable approximate group:

- we want to associate to it the coarse (equivalence) class $[\Lambda]_c$ of (mutually coarsely equivalent) metric spaces, and
- define $\operatorname{asdim} \Lambda$ to be $\operatorname{asdim}([\Lambda]_c)$, i.e., asdim of any metric space representing $[\Lambda]_c$.

Lemma

If G is a countable group, and $\Lambda \subseteq G$ is a <u>*subset*</u>*, and if we take any two left-invariant proper metrics d and d' on G, then $id : (\Lambda, d|_{\Lambda\times\Lambda}) \to (\Lambda, d'|_{\Lambda\times\Lambda})$ is a coarse equivalence.*

In particular, apply this on a countable approximate group $(\Lambda, \Lambda^\infty)$, (i.e., on $\Lambda \subseteq \Lambda^\infty$): take any left-invariant proper metric d on Λ^∞, define the (canonical) coarse class of Λ:

$$[\Lambda]_c := [(\Lambda, d|_{\Lambda\times\Lambda})]_c.$$

Note (independence of the ambient group): If Λ is an approximate subgroup of a countable group G, and if d is a left-invariant proper metric on G, then $d|_{\Lambda^\infty\times\Lambda^\infty}$ is a left-invariant proper metric on Λ^∞, so $[\Lambda]_c = [(\Lambda, (d|_{\Lambda^\infty\times\Lambda^\infty})|_{\Lambda\times\Lambda})]_c$ is independent of the ambient group which is used to define it.

Note: $[\Lambda]_c$ admits a representative which is a proper metric space.

Finally, for a countable approximate group $(\Lambda, \Lambda^\infty)$, define

$$\operatorname{asdim} \Lambda := \operatorname{asdim}([\Lambda]_c)$$

$$= \operatorname{asdim} \text{ of any metric space representing } [\Lambda]_c.$$

Lemma

If $(\Lambda, \Lambda^\infty)$ is a countable approximate group, then $\forall k \in \mathbb{N}$, the inclusion $\Lambda \hookrightarrow \Lambda^k$ is a coarse equivalence, so $[\Lambda]_c = [\Lambda^k]_c$.

Corollary

If $(\Lambda, \Lambda^\infty)$ is a countable approximate group, then $\operatorname{asdim} \Lambda \leq \operatorname{asdim} \Lambda^\infty$, and $\operatorname{asdim} \Lambda^k = \operatorname{asdim} \Lambda$, $\forall k \in \mathbb{N}$.

A theorem on asdim of approximate groups

and the theorem which inspired it.

Theorem (Buyalo-Lebedeva, 2007)

For a hyperbolic group G, $\operatorname{asdim} G = \dim \partial G + 1$.
In fact, this is true for proper, geodesic, Gromov hyperbolic, cobounded metric spaces.

For approximate groups:

Theorem (Cordes-Hartnick-T.)

For a hyperbolic approximate group $(\Lambda, \Lambda^\infty)$,

$$\operatorname{asdim} \Lambda = \dim \partial \Lambda + 1.$$

In fact, this is true for proper, geodesic, Gromov hyperbolic, quasi-cobounded metric spaces.

Theorem we wish to generalize

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Theorem we wish to generalize

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In fact, this is true for proper, geodesic, Gromov hyperbolic, cobounded metric spaces.

We should recall and/or define:

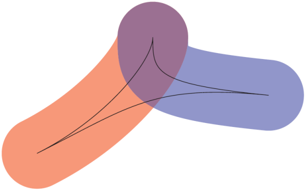
- the notion of being (Gromov) hyperbolic for a (nice enough) metric space, group, approximate group,
- (Gromov) boundary for a (nice enough) hyperbolic space,
- properness, coboundedness and quasi-coboundedness.

Definition

A metric space is proper if all closed balls in it are compact.

Definition

A geodesic metric space is called *(Gromov) hyperbolic* if $\exists\delta \geq 0$ such that all geodesic triangles are δ-thin, i.e., every side of a geodesic triangle is contained in δ-nbhd of the union of the other two sides.



This is also called being δ-hyperbolic. Let us agree that a 0-nbhd of a triangle = the triangle, so a tripod Y in a graph is 0-hyperbolic.

Theorem

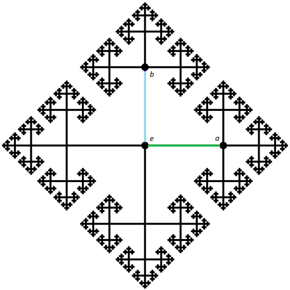
(Gromov) hyperbolicity is a QI invariant for geodesic metric spaces.

Gromov hyperbolic spaces (and groups)

This definition generalizes the metric properties of classical hyperbolic geometry and of (graphs that are) trees.

- Some examples:
  1. hyperbolic plane $\mathbb{H}^2$ (also $\mathbb{H}^n$, $\forall n \in \mathbb{N}_{\geq 2}$),
  2. any bounded metric space,
  3. hyperbolic groups (finitely generated groups G with Cayley graph $\Gamma_S(G)$ (Gromov) hyperbolic) ...

for 3 in particular: Cayley graph $\Gamma_{\{a,b,a^{-1},b^{-1}\}}(F_2)$ of the free group of rank 2:

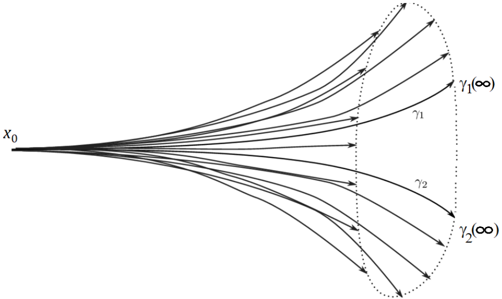


Gromov boundary

Definition

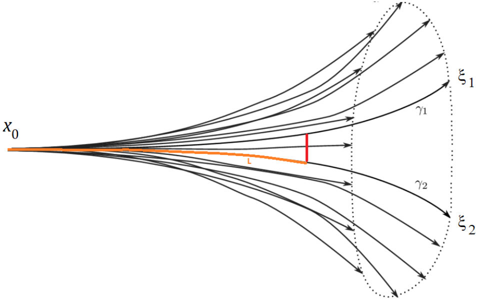
For a proper geodesic (Gromov) hyperbolic space X, its *(Gromov) boundary* ∂X consists of points that are equivalence classes of geodesic rays in X, where two geodesic rays are equivalent if they fellow-travel, i.e., they are within finite Hausdorff distance from eachother ($\sup_{t\in[0,\infty)} d(\gamma(t), \gamma'(t)) < \infty$).

Elements of ∂X: $\gamma(\infty)$ or ξ.



Metric on ∂X: (vague definition a visual metric on ∂X)
For ξ_1, $\xi_2 \in \partial X$, and some fixed $x_0 \in X$, take a geodesic ray γ_1 from x_0 to ξ_1, and a geodesic ray γ_2 from x_0 to ξ_2. These will fellow-travel for some distance L, before they diverge. Define $\varrho(\xi_1, \xi_2) := e^{-L}$ (or $e^{-\varepsilon L}$, not a metric yet). Now if η_1, $\eta_2 \in \partial X$, put

$$d(\eta_1, \eta_2) := \inf \left\{ \sum_{i=1}^{n} \varrho(\xi_{i-1}, \xi_i) \mid \eta_1 = \xi_0, \ldots, \xi_n = \eta_2,\ n \in \mathbb{N} \right\}.$$



Some properties (for X proper geodesic hyperbolic):

- two visual metrics on ∂X induce the same topology on ∂X,
- $(\partial X, d)$ is bounded, complete, compact (for d any visual metric).

Theorem

If (X, d_X), (Y, d_Y) are two proper geodesic hyperbolic spaces which are quasi-isometric, then ∂X and ∂Y are homeomorphic.

Some examples:

- $\partial\mathbb{H}^2 \approx S^1 \quad (\partial\mathbb{H}^n \approx S^{n-1})$
- $\partial(\Gamma_{\{a,b,a^{-1},b^{-1}\}}(F_2)) \approx$ Cantor set.

Definition

A finitely generated group G is called hyperbolic if for any finite generating set S of G, the Cayley graph $\Gamma_S(G)$ is a hyperbolic metric space.

Note that we know that:

- Cayley graphs of fin.gen. groups are geodesic metric spaces (with path-length metrics, i.e., word metrics d_S),
- for S and S' finite generating sets of G, we have $(\Gamma_S(G), d_S) \overset{QI}{\approx} (\Gamma_{S'}(G), d_{S'})$,
- hyperbolicity is a QI invariant of geodesic metric spaces

$\Rightarrow$ hyperbolicity of finitely generated groups is well-defined.

- Cayley graphs of fin.gen. groups are proper and geodesic, so if G hyperbolic, define $\partial G := \partial(\Gamma_S(G), d_S)$.

Gromov hyperbolic spaces (and groups)

Some examples:

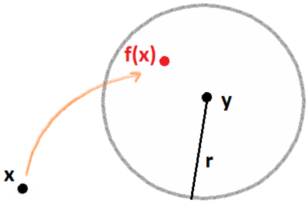
- Elementary hyperbolic groups:
  - finite groups ($\Rightarrow$ Cayley graph of finite diameter),
  - $\mathbb{Z}$ and virtually cyclic groups (containing $\mathbb{Z}$ as a finite index subgroup)
- finitely generated free groups,
- small cancellation groups,
- fundamental groups of closed surfaces with genus > 1,
- fundamental groups of closed, negatively curved manifolds.

Non-examples:

- $\mathbb{Z}^2$ ($\overset{QI}{\approx}$ $(\mathbb{R}^2, d_E)$),
- any group containing $\mathbb{Z}^2$ as a subgroup,
- Baumslag–Solitar groups $B(m, n)$.

Coboundedness and quasi-coboundedness

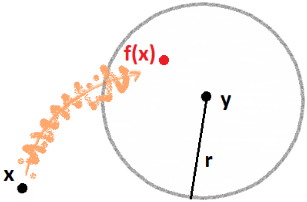
A metric space (X, d) is said to be *cobounded* if there is an $r > 0$ so that for all $x, y \in X$ there is an isometry $f : X \to X$ so that $d(f(x), y) < r$.



Or, equivalently, there exists a bounded subset A of X s.t. the orbit of A, under the $Isometry(X)$ acting on X, covers X.

Coboundedness and quasi-coboundedness

For a metric space (X, d) and for $K \geq 1$, $C \geq 0$, $r > 0$, we say that X is (K, C, r)-quasi-cobounded if for all $x, y \in X$ there is a (K, C, C)-quasi-isometry $f : X \to X$ such that $d(f(x), y) < r$.



(X, d) is quasi-cobounded if it is (K, C, r)-quasi-cobounded, for some K, C, r as above.
(Note that X is cobounded if it is $(1, 0, 0)$-quasi-cobounded (those maps f are isometries).

Approximate groups – before introducing hyperbolicity

Recall that we have defined hyperbolicity for finitely generated groups. How does this translate to approximate groups?

For a group: being finitely generated

$\downarrow$

For an approximate group:

- being algebraically finitely generated
- being geometrically finitely generated

Definition

For $(\Lambda, \Lambda^{\infty})$ we say it is algebraically finitely generated if Λ^{∞} is a finitely generated group.

Definition

A countable approximate group $(\Lambda, \Lambda^\infty)$ is said to be geometrically finitely generated if $(\Lambda, d|_{\Lambda\times\Lambda})$ is coarsely connected, where d is a left-invariant proper metric on Λ^∞.

Coarsely connected = connected by "coarse paths": $\exists c > 0$ s.t. $\forall x, x' \in \Lambda$, there is a *c-path* from x to x', i.e., $\exists$ a finite sequence $x = x_0, x_1, \ldots, x_{n-1}, x_n = x'$ in Λ so that $d(x_i, x_{i+1}) < c$, for $i = 0, \ldots .n - 1$.

(For a countable approximate group, being geometrically finitely generated $\Rightarrow$ being algebraically finitely generated. But not the other way around.)

Approximate groups – before introducing hyperbolicity

Theorem

Let $(\Lambda, \Lambda^\infty)$ be a countable approximate group, and let d be a left-invariant proper metric on Λ^∞. Then: $(\Lambda, d|_{\Lambda \times \Lambda})$ is coarsely connected $\Leftrightarrow$ there is a representative $X \in [\Lambda]_c$ which is large-scale geodesic.

Large-scale geodesic means: $\exists a > 0, b \geq 0, c > 0$ such that $\forall x, x' \in X$ there is a c-path between x, x' of length$\leq a \cdot d(x, x') + b$.

Now, for $(\Lambda, \Lambda^\infty)$ geometrically finitely generated, we define the internal QI type of $(\Lambda, \Lambda^\infty)$:

$$[\Lambda]_{\mathrm{int}} := \{X \in [\Lambda]_c \mid X \text{ large-scale geodesic}\},$$

Note:
- X large-scale geodesic $\Leftrightarrow$ $X \overset{QI}{\approx}$ to a geodesic metric space,
- For $X, X' \in [\Lambda]_{\mathrm{int}}$, we have $X \overset{QI}{\approx} X'$.

Note that, for $(\Lambda, \Lambda^\infty)$ geometrically finitely generated:

- $[\Lambda]_{\mathrm{int}}$ can always be represented by a proper metric d on Λ, called *internal metric* on Λ ("large-scale path metric").
- For internal metric d, (Λ, d) is proper and large-scale geodesic, so $(\Lambda, d) \overset{QI}{\approx}$ to a locally finite graph X_Λ, which we call a generalized Cayley graph of $(\Lambda, \Lambda^\infty)$.
- we can choose a representative (X, d) of $[\Lambda]_{\mathrm{int}} \subseteq [\Lambda]_c$ which is a *proper, geodesic and quasi-cobounded metric space*. We will call such a space an apogee for $(\Lambda, \Lambda^\infty)$.

Hyperbolic approximate groups

Recall the definition for *groups*: A finitely generated group G is *hyperbolic* if one (hence any) Cayley graph $\Gamma_S(G)$ of it (with respect to a finite generating set S) is (Gromov) hyperbolic.

Definition (Hyperbolicity for approximate groups)

A geometrically finitely generated approximate group $(\Lambda, \Lambda^\infty)$ is said to be hyperbolic if one (hence any) apogee of it is hyperbolic. Equivalently, if some (hence any) generalized Cayley graph of it is hyperbolic.

Note: For a hyperbolic approximate group $(\Lambda, \Lambda^\infty)$, an apogee $(X, d) \in [\Lambda]_{\mathrm{int}} \subseteq [\Lambda]_c$ is a proper geodesic hyperbolic quasi-cobounded space.

B.-L. Theorem for hyperbolic approximate groups

Theorem (Cordes-Hartnick-T.)

For a hyperbolic approximate group $(\Lambda, \Lambda^\infty)$*,*

$$\operatorname{asdim} \Lambda = \dim \partial\Lambda + 1.$$

In fact, this is true for proper geodesic hyperbolic quasi-cobounded metric spaces.

How do we define the (Gromov) boundary $\partial\Lambda$:

- take any apogee $(X, d) \in [\Lambda]_{\mathrm{int}} \subseteq [\Lambda]_c$,
- recall that, if (X, d_X), (Y, d_Y) are proper geodesic hyperbolic spaces s.t. $X \overset{QI}{\approx} Y$, then $\partial X \approx \partial Y$,
- therefore define $\partial\Lambda := [\partial X]_{homeo}$ = the homeomorphism class of ∂X, for any apogee $(X, d) \in [\Lambda]_{\mathrm{int}}$,
- recall that $\dim$ is a topological invariant (i.e., preserved by homeomorphisms).

B.-L. Theorem for hyperbolic approximate groups

Equivalently, the first part of this theorem is saying:

Theorem

For any apogee X of a hyperbolic approximate group $(\Lambda, \Lambda^\infty)$, we have

$$\operatorname{asdim} X = \dim \partial X + 1.$$

In full generality, the theorem we prove is:

Theorem

For a metric space X which is proper, geodesic, hyperbolic and quasi-cobounded, we have

$$\operatorname{asdim} X = \ell\text{-}\dim(\partial X, d) + 1 = \dim \partial X + 1,$$

where d is any visual metric on ∂X.

Here ℓ- dim denotes *linearly controlled metric dimension*.

Outline of the proof

We need to show:

- $\operatorname{asdim} X \geq \dim \partial X + 1$ and • $\operatorname{asdim} X \leq \dim \partial X + 1$.

The first of these two inequalities works without the assumption of coboundedness or quasi-coboundedness:

Theorem (Buyalo-Schroeder)

If X is a proper, geodesic, hyperbolic metric space, then

$$\operatorname{asdim} X \geq \dim \partial X + 1.$$

This is not too hard to prove, using a hyperbolic cone of ∂X and its embedding into X, and then some properties of $\dim \ldots$

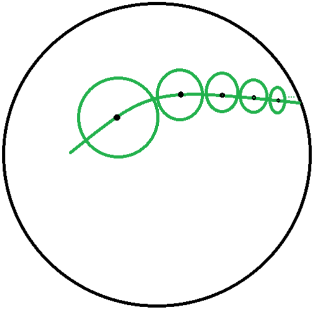
Note that equality holds when X is a bounded metric space, since $\operatorname{asdim} X = 0$ and $\dim \partial X = \dim \emptyset = -1$.

But if X is unbounded, "$\leq$" does not work with only the assumptions from B.-S. Theorem, as shown in the following example:

Outline of the proof

Example (hyperbolic shish-kebab (or shashlik or skewer)):
Let $n \geq 2$, $\gamma : [0, \infty) \to \mathbb{H}^n$ be a geodesic ray, and let $x_1, x_2, \ldots$ be points on $\gamma([0,\infty))$ such that $d(x_k, x_{k+1}) \geq 2^{k+2}, \forall k \in \mathbb{N}$. Define

$$X = \gamma([0,\infty)) \cup \bigcup_{k \in \mathbb{N}} B(x_k, 2^k) \ \subset \mathbb{H}^n.$$



With path-lenght metric, X is a proper geodesic hyperbolic space, which contains arbitrarily large balls of $\mathbb{H}^n$, so $\operatorname{asdim} X = n$.

But X contains a single geodesic ray, so ∂X is just one point $\Rightarrow$ $\dim \partial X = 0$.
So $\operatorname{asdim} X \not\leq \dim \partial X + 1$, since $n \not\leq 0 + 1$.

Outline of the proof

Let us list the main steps of the proof for

$$\operatorname{asdim} X \leq \ell\text{-}\dim(\partial X, d) + 1 \leq \dim \partial X + 1,$$

when X is an unbounded, proper, geodesic, hyperbolic and quasi-cobounded space, and d is any visual metric on ∂X.
First of all, the following lemmas are true [Cordes-Hartnick-T.]:

L1: X is a visual space (has coarse version of the geodesic extension property)

L2: $(\partial X, d)$ is locally quasi-similar to itself (i.e., there are constants $\lambda \geq 1$, $K \geq 1$, and $R_0 > 1$ s.t. $\forall R > R_0$ and $\forall C \subset \partial X$ with $\operatorname{diam} C \leq \frac{1}{R}$, $\exists$ a map $f : C \to \partial X$ such that $\forall x_1, x_2 \in C$
$\frac{1}{\lambda} R^K (d(x_1, x_2))^K \ \leq \ d(f(x_1), f(x_2)) \ \leq \ \lambda \sqrt[K]{R} \sqrt[K]{d(x_1, x_2)}$.)

L3: $(\partial X, d)$ is doubling, i.e., $\exists N \in \mathbb{N}$ s.t. for all $t > 0$ and all $\xi \in \partial X$ there exist $\xi_1, \ldots, \xi_N \in \partial X$ s.t. $B(\xi, 2t) \subset \bigcup_{i=1}^{N} B(\xi_i, t)$.

Now we use the following:

Thm1: [Buyalo-Schroeder] Since $(\partial X, d)$ is doubling (at small scales), then $\ell\text{-}\dim(\partial X, d) < \infty$.

Thm2: [Buyalo-Schroeder] Any visual hyperbolic space X with $\ell\text{-}\dim(\partial X, d) = n$ can be QI-embedded into the product of $n+1$ simplicial trees, i.e., $\exists\ X \overset{QI}{\hookrightarrow} T_1 \times \ldots \times T_{n+1}$.

Cor: We know that $\operatorname{asdim} T_i \leq 1$, so $\operatorname{asdim}(T_1 \times \ldots \times T_{n+1}) \leq n+1$, by the product theorem for asdim.
Therefore $\operatorname{asdim} X \leq n+1 = \ell\text{-}\dim(\partial X, d) + 1$.

Thm3 [C.-H.-T.] If a metric space $(\partial X, d)$ is locally quasi-similar to itself, and $\ell\text{-}\dim(\partial X, d) < \infty$, then $\ell\text{-}\dim(\partial X, d) \leq \dim \partial X$.

Prop: In general, for a metric space (Z, d): $\ell\text{-}\dim(Z, d) \geq \dim Z$.

So

$$\operatorname{asdim} X \leq \ell\text{-}\dim(\partial X, d) + 1 = \dim \partial X + 1.$$

Importance of formulas like $\operatorname{asdim} X = \dim \partial X + 1$

- For hyperbolic groups, $\operatorname{asdim} G = \dim \partial G + 1$ means that using $\dim$ (of the boundary) we can estabish the finiteness of asdim of the group, and groups with finite asdim are important, for example, for Novikov's conjecture (in topology of manifolds).
- For hyperbolic approximate groups, $\operatorname{asdim} \Lambda = \dim \partial \Lambda + 1$ is useful in proving some other interesting facts, like the fact that every non-elementary hyperbolic approximate group of $\operatorname{asdim} = 1$ is QI to a fin. generated, non-abelian free group.

Literature and authors

Matthew Cordes, Tobias Hartnick, and Vera Tonić.
Foundations of geometric approximate group theory.
https://arxiv.org/pdf/2012.15303.pdf.

Thank you!