Groupoid C^*-algebras in solid state physics

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Based on joint work with:

Emil Prodan (Yeshiva University):

- B. Mesland and E. Prodan, *Classifying the Dynamics of Architected Materials by Groupoid Methods*, arXiv:2308.07866
- B. Mesland and E. Prodan, *A groupoid approach to interacting fermions*, Commun. Math. Phys. 394, 143–213 (2022)

Chris Bourne (Nagoya University):

- C. Bourne and B. Mesland, *Localised module frames and Wannier bases from groupoid Morita equivalences*, J. Fourier Anal. Appl. (27), Art nr 69 (2021).
- C. Bourne and B. Mesland, *Index theory and topological phases of aperiodic lattices*, Ann. Henri Poincaré 20, 1969–2038 (2019).

Also relevant:

- U. Enstad, S. Raum, *A dynamical approach to sampling and interpolation in unimodular groups*, arXiv:2207.05125 (2022).

Idea: use discrete point patterns to model solid materials

The points are the "building blocks" of the material, that may or may not have "internal structure".

Atoms: no internal structure.

Mechanical resonator (such as quartz): internal structure such as shape and orientation.

Both cases can be treated on equal footing by changing the ambient space of the pattern.

Mechanical resonator: confined mechanical system with finite set $\boldsymbol{q} = \{q_1, \ldots, q_N\}$ of degrees of freedom

We will be concerned with finite and infinite clusters of identical *seed* resonators located at a discrete subset $\mathcal{L}$ of "oriented points" in the physical space $\mathbb{E}^d$, $d = 1, 2, 3$. We thus view $\mathcal{L}$ as a subset of $\mathrm{Iso}(\mathbb{E}^d)$.

Quadratic Lagrangian:

$$L_{\mathcal{L}}(\dot{\boldsymbol{q}}, \boldsymbol{q}) = \frac{1}{2} \sum_{x \in \mathcal{L}} \dot{\boldsymbol{q}}_x \cdot \widehat{M}_0 \cdot \dot{\boldsymbol{q}}_x^T - \frac{1}{2} \sum_{x, x' \in \mathcal{L}} \boldsymbol{q}_x \cdot \widehat{W}_{x,x'}(\mathcal{L}) \cdot \boldsymbol{q}_{x'}^T,$$

Degrees of freedom observed and quantified using equipment that is rigidly attached to the frame of the resonator.

The numerical values $\{q_1, \ldots, q_N\}$ recorded by the local research assistants are not affected by translations, rotations or reflections of the resonators.

Physics encoded in the relation

$$\mathcal{L} \mapsto L_{\mathcal{L}}(\dot{\boldsymbol{q}}_1, \ldots, \dot{\boldsymbol{q}}_r, \boldsymbol{q}_1, \ldots, \boldsymbol{q}_r), \quad \dot{\boldsymbol{q}}_i, \ \boldsymbol{q}_i \in \mathbb{R}^N.$$

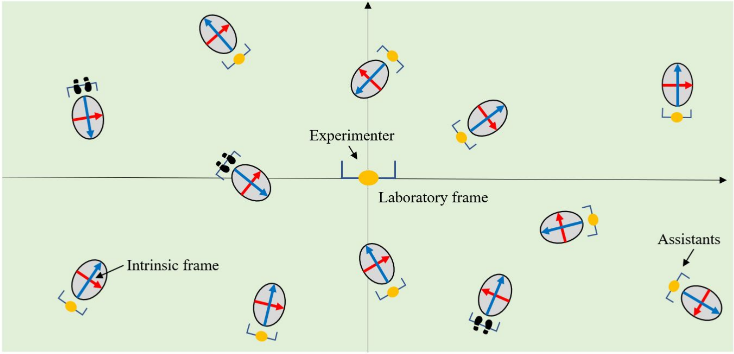

Figure: Example of a physical system addressed in this work. It consist of identical resonators placed at selected points of the plane and with selected orientations. Assistants observe and quantify the dynamics of the local degrees of freedom using a local frame rigidly attached to the resonators.

Dynamical matrix

Quadratic Lagrangian:

$$L_{\mathcal{L}}(\dot{\boldsymbol{q}}, \boldsymbol{q}) = \frac{1}{2} \sum_{x \in \mathcal{L}} \dot{\boldsymbol{q}}_x \cdot \widehat{M}_0 \cdot \dot{\boldsymbol{q}}_x^T - \frac{1}{2} \sum_{x,x' \in \mathcal{L}} \boldsymbol{q}_x \cdot \widehat{W}_{x,x'}(\mathcal{L}) \cdot \boldsymbol{q}_{x'}^T,$$

We encode the dynamics the *dynamical matrix*:

$$D_{\mathcal{L}} = \sum_{x,x' \in \mathcal{L}} |x\rangle \otimes w_{x,x'}(\mathcal{L}) \otimes \langle x'| \in \mathcal{B}(\ell^2(\mathcal{L}, \mathbb{C}^N)),$$

with

$$w_{x,x'}(\mathcal{L}) = \widehat{M}_0^{-\frac{1}{2}} \, \widehat{W}_{x,x'}(\mathcal{L}) \, \widehat{M}_0^{-\frac{1}{2}} \in M_N(\mathbb{C}).$$

In the presence of Galilean invariance, we have the equivariance relation

$$w_{g \cdot x, g \cdot x'}(g \cdot \mathcal{L}) = w_{x,x'}(\mathcal{L}), \quad g \in \mathrm{Iso}(\mathbb{E}^d),$$

which can be used to reduce the expression of the dynamical matrix

$$D_{\mathcal{L}} = \sum_{x,x' \in \mathcal{L}} |x\rangle \otimes w_{e, x \cdot x'}(x \cdot \mathcal{L}) \otimes \langle x'|.$$

This shows that the entire dynamics is encoded in the $M_N(\mathbb{C})$-valued map $(g, \mathcal{L}) \mapsto w_{e,g}(\mathcal{L})$, defined on tuples $(g, \mathcal{L})$ with $g \in \mathcal{L}$.

For precision measurements one needs control over the dynamical matrices:

- Continuity: the map $(x, x', \mathcal{L}) \mapsto w_{x,x'}(\mathcal{L})$ is continuous is an appropriate sense;
- Equivariance: $w_{gx,gx'}(g\mathcal{L}) = w_{x,x'}(\mathcal{L})$;
- Finite coupling range: $w_{x,x'}(\mathcal{L}) = 0$ whenever $x' \cdot x$ is outside a compact set.

Claim: such matrices can be measured in a laboratory. Equivariance assures us that it is enough to focus the observations only on x, x' in a compact vicinity of the origin. By continuitya good approximation of the map $\mathcal{L} \to w_{x,x'}(\mathcal{L})$ can be derived by interpolating a finite sampling of $\mathcal{L}$'s. This assures us that, even for an infinite architecture, we still have a chance to measure with enough precision every single coupling matrix of the system.

Goal: find the smallest C^*-algebra that contains the dynamical matrices.

G locally compact Hausdorff topological group, $\mathcal{L} \subset G$ closed subset;

$\mathcal{L}$ is a *Delone set* if:

- there is a nonempty open set $U \subset G$ such that for all $g \in G$: $|gU \cap \mathcal{L}| \leq 1$ (U-separated);
- there is a compact set $K \subset G$ such that for all $g \in G$: $|gK \cap \mathcal{L}| \geq 1$ (K-dense);

Classical Delone sets $G = \mathbb{R}^d$:

An (r, R)-Delone set $\mathcal{L} \subset \mathbb{R}^d$:

- *uniformly r-discrete*: $\forall x \in \mathbb{R}^d : |B(x, r) \cap \mathcal{L}| \leq 1$,
- *relatively R-dense*: $\forall x \in \mathbb{R}^d : |\overline{B(x, R)} \cap \mathcal{L})| \geq 1$.

Important: we do not assume any translational symmetry for $\mathcal{L}$.

The space of $\mathcal{C}(G)$ of closed subsets of G carries the *Fell topology*. For fixed U, the subset of all U-seperated sets is closed. To describe the physics in a translation invariant way we enlarge our space to obtain an G-action. Consider the *hull* of $\mathcal{L}$:

$$\Omega_{\mathcal{L}} := \overline{\{g \cdot \mathcal{L} : g \in G\}}.$$

For $\mathcal{L} = \mathbb{Z}^d$ we have $\Omega_{\mathbb{Z}^d} = \mathbb{T}^d$ but in general it can be quite complicated.

We obtain a continuous crossed product C^*-algebra $C(\Omega_{\mathcal{L}}) \rtimes G$, which contains a larger class of dynamical matrices than we are after. To make it smaller, we need the notion of a *groupoid*.

Groupoids form a class of mathematical objects unifying spaces, groups and equivalence relations.

They also form a bridge between commutative and noncommutative topology/geometry

Groupoids arise in many different contexts: dynamics, foliations, number theory and physics.

Definition

A groupoid is a small category in which all morphisms are invertible. More concretely it is a set $\mathcal{G}$ with a subset $\mathcal{G}^{(2)} \subset \mathcal{G} \times \mathcal{G}$, a multiplication map $\mathcal{G}^{(2)} \to \mathcal{G}$, $(\gamma, \xi) \mapsto \gamma\xi$ and an inverse $\mathcal{G} \to \mathcal{G}$ $\gamma \mapsto \gamma^{-1}$ such that

(i) $(\gamma^{-1})^{-1} = \gamma$ for all $\gamma \in \mathcal{G}$,

(ii) if $(\gamma, \xi), (\xi, \eta) \in \mathcal{G}^{(2)}$, then $(\gamma\xi, \eta), (\gamma, \xi\eta) \in \mathcal{G}^{(2)}$,

(iii) $(\gamma, \gamma^{-1}) \in \mathcal{G}^{(2)}$ for all $\gamma \in \mathcal{G}$,

(iv) for all $(\gamma, \xi) \in \mathcal{G}^{(2)}$, $(\gamma\xi)\xi^{-1} = \gamma$ and $\gamma^{-1}(\gamma\xi) = \xi$.

Given a groupoid we denote by $\mathcal{G}^{(0)} = \{\gamma\gamma^{-1} : \gamma \in \mathcal{G}\}$ the space of units and define the source and range maps $r, s : \mathcal{G} \to \mathcal{G}^{(0)}$ by the equations

$$r(\gamma) = \gamma\gamma^{-1}, \qquad s(\gamma) = \gamma^{-1}\gamma$$

for all $\gamma \in \mathcal{G}$. The source and range maps allow us to characterise

$$\mathcal{G}^{(2)} = \big\{(\gamma, \xi) \in \mathcal{G} \times \mathcal{G} : s(\gamma) = r(\xi)\big\}.$$

A *topological groupoid* is a groupoid $\mathcal{G}$ equipped with second countable, locally compact and Hausdorff topology such that the mulitplication, inversion, source and range maps are all continuous.

A groupoid $\mathcal{G}$ is étale if the range map $r : \mathcal{G} \to \mathcal{G}^{(0)}$ is a local homeomorphism. Étale groupoids have the useful property that for all $x \in \mathcal{G}^{(0)}$, the fibres $r^{-1}(x)$ and $s^{-1}(x)$ are discrete.

There is a notion of Morita equivalence of groupoids. This roughly entails the existence of a space that admits commuting left and right actions that are free and proper in the appropriate sense. Morita equivalence is an equivalence relation on groupoids.

Examples of groupoids:

- A locally compact space X is a groupoid with $\mathcal{G} = \mathcal{G}^{(0)} = X$;
- A locally compact group G is a groupoid with $\mathcal{G} = G$, $\mathcal{G}^{(0)} = \{e\}$ and inversion and composition determined by the group operation;
- Let $\sim$ be an equivalence relation on X. Then its *graph* defines a groupoid

$$\mathcal{G} := \{(x_1, x_2) : x_1 \sim x_2\}, \quad \mathcal{G}^{(0)} := \{(x, x)\}$$

  with

$$r(x_1, x_2) = x_1, \quad s(x_1, x_2) = x_2, \quad (x_1, x_2)(x_2, x_3) = (x_1, x_3), \quad (x_1, x_2)^{-1} = (x_2, x_1)$$

- Let G be a locally compact group acting on X by homeomorphisms. Then the Cartesian product $G \rtimes X$ is a groupoid with

$$r(g, x) = gx, \quad s(g, x) = x, \quad (h, gx)(g, x) := (hg, x), \quad (g, x)^{-1} := (g^{-1}, gx).$$

  Morita equivalent to $G \setminus X$ if the action is free and proper.

Definition

A topological groupoid $\mathcal{G}$ admits an abstract transversal if there is a closed subset $X \subset \mathcal{G}^{(0)}$ such that

- X meets every orbit of the $\mathcal{G}$-action on $\mathcal{G}^{(0)}$;
- for the relative topologies on X and

$$\mathcal{G}_X := \{\gamma \in \mathcal{G} : s(\gamma) \in X\} \subset \mathcal{G},$$

  the restrictions $r : \mathcal{G}_X \to \mathcal{G}^{(0)}$ and $s : \mathcal{G}_X \to X$ are open maps.

Given an abstract transversal $X \subset \mathcal{G}^{(0)}$,

$$\mathcal{G} \xleftarrow{r} \mathcal{G}_X \xrightarrow{s} \mathcal{H}$$

is a $\mathcal{G}$–$\mathcal{H}$ groupoid equivalence for $\mathcal{H} = \{\gamma \in \mathcal{G}_X \, : \, r(\gamma) \in X\}$.

In examples, a non-étale groupoid $\mathcal{G}$ often admits a transversal X for which the groupoid $\mathcal{H}$ is étale. Examples of this include transitive groupoids, groupoids from foliations, Smale spaces, and Delone sets.

The continuous groupoid $\Omega_{\mathcal{L}} \rtimes G$ admits a *transversal*

$$\Omega_e := \{\omega \in \Omega_{\mathcal{L}} : e \in \omega\}.$$

Proposition (Enstad-Raum)

If $\mathcal{L}$ is uniformly separated then the space Ω_e is an abstract transversal for $\Omega_{\mathcal{L}} \rtimes G$.

Examples:

- For $\mathcal{L} = \mathbb{Z}^d \subset \mathbb{R}^d$ the tranversal is a single point.
- For the pattern $\{n + \lambda(\xi_n - \frac{1}{2})\}_{n\in\mathbb{Z}}$, with the ξ_n entries drawn randomly and distinctly from the interval $[0, 1]$ and $\lambda < 1$, the transversal is the Hilbert cube $[0, 1]^{\mathbb{Z}}$.

The groupoid

$$\mathcal{G}_{\mathcal{L}} := \{(x,\omega) \in G \rtimes \Omega_0 : x \in \omega\} \subset G \rtimes \Omega_0$$

is an étale groupoid with compact unit space Ω_0.

The groupoid C^*-algebras of $G \rtimes \Omega_{\mathcal{L}}$ and $\mathcal{G}$ are Morita equivalent.

In $G \rtimes \Omega_{\mathcal{L}}$ the fibers of the range and source maps $r, s : G \rtimes \Omega_{\mathcal{L}} \to \Omega_{\mathcal{L}}$ are in bijection with G.

In $\mathcal{G}_{\mathcal{L}}$ the fibers of the range and source maps are in bijection with $\mathcal{L}$ (and its translates).

We briefly review the construction of groupoid C^*-algebras.

Definition

A Haar system on a locally compact Hausdorff groupoid $\mathcal{G}$ is a set of measures $\{\nu^x : x \in \mathcal{G}^{(0)}\}$ on $\mathcal{G}$ such that $\mathrm{supp}(\nu^x) = r^{-1}(x)$ and for all $f \in C_c(\mathcal{G})$,

$$\int_{\mathcal{G}} f(\xi)\,\mathrm{d}\nu^{r(\eta)}(\xi) = \int_{\mathcal{G}} f(\eta\xi)\,\mathrm{d}\nu^{s(\eta)}(\xi), \qquad g(x) := \int_{\mathcal{G}} f(\xi)\,\mathrm{d}\nu^{x}(\xi) \in C(\mathcal{G}^{(0)}).$$

- Étale groupoids always have a Haar system given by the counting measure on the (discrete) fibres $r^{-1}(x)$.
- A transformation groupoid $G \rtimes X$ carries a Haar system induced by the Haar measure on G.

Given $\mathcal{G}$ with Haar system $\{\nu^x\}_{x\in\mathcal{G}^{(0)}}$, we can define a $*$-algebra structure on $C_c(\mathcal{G})$:

$$(f_1 * f_2)(\eta) = \int_{\mathcal{G}} f(\xi)g(\xi^{-1}\eta)\,\sigma(\xi,\xi^{-1}\eta)\,\mathrm{d}\nu^{r(\eta)}(\xi), \qquad f^*(\xi) = \overline{f(\xi^{-1})},$$

as well as a a family of inner products parametrised by $x \in \mathcal{G}^{(0)}$:

$$\langle f, g\rangle_x := \int \overline{f(\xi^{-1})}g(\xi^{-1})\mathrm{d}\nu^x\xi.$$

This gives:

- A family of *regular representations* π_x of the $*$-algebra $C_c(\mathcal{G})$ on $L^2(\mathcal{G},\nu^x)_{x\in\mathcal{G}^{(0)}}$ by convolution;
- the closure of $C_c(\mathcal{G})$ in the norm $\sup_{x\in\mathcal{G}^{(0)}}\|\pi_x(f)\|$ gives the *reduced groupoid C^*-algebra* $C_r^*(\mathcal{G})$.
- Morita equivalent groupoids give rise to Morita equivalent C^*-algebras;
- for any C^*-algebra we obtain $C_r^*(\mathcal{G},A) = C_r^*(\mathcal{G}) \otimes A$.

For a pattern $\mathcal{L} \subset G$ the left regular representations of $C^*(\mathcal{G}_{\mathcal{L}}, M_n(\mathbb{C}))$ are supported on

$$\mathcal{H}_{\mathcal{L}} := \ell^2\left(\mathcal{L}, \mathbb{C}^N\right),$$

and the representation acts explicitly as

$$[\pi_{\mathcal{L}}(f)\varphi](g') = \sum_{g \in \mathcal{L}} f\left(g \cdot (g', \mathcal{L})\right) \cdot \varphi(g), \quad g' \in \mathcal{L},\ \varphi \in \ell^2\left(\mathcal{L}, \mathbb{C}^N\right).$$

In particular one verifies that

$$\pi_{\mathcal{L}}(f)(|g'\rangle \otimes \alpha) = \sum_{g \in \mathcal{L}} f\left(g \cdot (g', \mathcal{L})\right) \cdot |g\rangle \otimes \alpha, \quad \alpha \in \mathbb{C}^N.$$

Recall the expression of a generic dynamical matrix

$$D_{\mathcal{L}} = \sum_{x, x' \in \mathcal{L}} |x\rangle \otimes w_{e, x \cdot x'}(x \cdot \mathcal{L}) \otimes \langle x'|$$

and, obviously,

$$D_{\mathcal{L}}(\alpha \otimes |x'\rangle) = \sum_{x \in \mathcal{L}} w_{e, x \cdot x'}(x \cdot \mathcal{L}) \cdot \alpha \otimes |x\rangle.$$

The transversal groupoid of a pattern reproduces the dynamical matrices.

Under the "physicality" assumptions, it is the smallest C^*-algebra that contains all of them.

- For systems with internal structure we use patterns in $\mathrm{Iso}(\mathbb{R}^d)$.
- For systems without internal structure we use Delone sets in $\mathbb{R}^d$

The K-theory and representation theory of such configurations are controlled by Morita equivalence classes of patterns, or rather, their groupoids.

In the case of a Delone set in $G = \mathbb{R}^d$ the groupoid $\mathcal{G}_{\mathcal{L}}$ can be used to model the dynamics of a *single* fermion hopping on $\mathcal{L}$.

A 2-cocycle $\sigma : \mathcal{G} \times \mathcal{G} \to S^1$ can be used to model the presence of a magnetic field.

Observables: the twisted groupoid C^*-algebra $C^*(\mathcal{G}, \sigma)$.

Theorem (Bellissard-Kellendonk)

The C^*-algebra $C^*(\mathcal{G}, \sigma)$ contains all Galilean invariant Hamiltonians for the dynamics of a single fermion.

Proposition (Freed–Moore, Thiang, Bourne–Carey–Rennie, Kellendonk, Kubota)

Suppose that $h = h^* \in C^*_r(\mathcal{G}, \sigma)$ has a spectral gap. Then h determines a class in $K_0(C^*_r(\mathcal{G}, \sigma))$. If h has a chiral symmetry, then h determines a class in $K_1(C^*_r(\mathcal{G}, \sigma))$.

Numerical invariants arise via

$$K_*(C^*_r(\mathcal{G}, \sigma)) \times KK^*(C^*_r(\mathcal{G}, \sigma), C(\Omega_0)) \to K_0(C(\Omega_0)) \xrightarrow{\int} \mathbb{C}.$$

Here $\int$ is the integral associated to a translation invariant measure on Ω_0.

To obtain a class in $KK_d(C^*(\mathcal{G},\sigma), C(\Omega_0))$ we consider

- The coordinate functions $X_k : \mathcal{G} \to \mathbb{R}, \quad (\omega, x) \mapsto x_k$
- The spinor bundle S_d of $\mathbb{R}^d$
- $D := \sum_{k=1}^{d} \gamma_k X_k : C_c(\mathcal{G}, S_d) \to C_c(\mathcal{G}, S_d)$ extends to a KK-cycle (E, D)

The class $[(E,D)] \in KK_d(C^*(\mathcal{G},\sigma), C(\Omega_0))$ gives the map

$$K_d(C^*(\mathcal{G},\sigma)) \xrightarrow{\otimes[D]} K_0(C(\Omega_0))$$

Invariant measures on Ω_0 give maps $K_0(C(\Omega_0)) \to \mathbb{C}$ and invariants of $C^*(\mathcal{G},\sigma)$.

Restriction to boundary of the material: dropping the coordinate x_d.

Consider

- The subgroupoid $\mathcal{H} := \{(\omega, x) \in \mathcal{G} : x_d = 0\} \subset \mathcal{G}$, and $C^*(\mathcal{H}, \sigma)$,
- The function $X_d : C_c(\mathcal{G}) \to C_c(\mathcal{G})$ extends to a KK-cycle (X, S).

The class $[(X, S)] \in KK_1(C^*(\mathcal{G}, \sigma), C^*(\mathcal{H}, \sigma))$ gives a map

$$K_d(C^*(\mathcal{G}, \sigma)) \xrightarrow{\otimes [S]} K_{d-1}(C^*(\mathcal{H}, \sigma)).$$

The boundary algebra $C^*(\mathcal{H}, \sigma)$ carries its own fundamental cycle (Y, T) built from $X_1, \cdots, X_{d-1}$ and the spinor bundle S_{d-1} on $\mathbb{R}^{d-1}$.

Theorem (Bourne–Mesland 2018)

There is a commutative diagram

$$\begin{array}{ccc} K_d(C^*(\mathcal{G},\sigma)) & \xrightarrow{\otimes X_d} & K_{d-1}(C^*(\mathcal{H},\sigma)) \\ \Big\downarrow \otimes D & & \Big\downarrow \otimes T \\ K_0(C(\Omega_0)) & = & K_0(C(\Omega_0)). \end{array}$$

Every K-theoretic invariant of the bulk algebra $C^*(\mathcal{G},\sigma)$ has a corresponding invariant in the edge algebra $C^*(\mathcal{H},\sigma)$. Their numerical invariants computed from the fundamental KK-class and an invariant measure on Ω_0 coincide.

To model N fermions we consider the topological space

$$\Omega_{\mathcal{L}}^{(N)} := \{(\omega, U, \chi) : \omega \in \Omega_{\mathcal{L}}, \quad U \subset \mathcal{L}_\omega, |U| = N, \quad \chi : \{1, \cdots, N\} \xrightarrow{\sim} U\}$$

The map

$$\Omega_{\mathcal{L}}^{(N)} \to \Omega_{\mathcal{L}}, \quad (\omega, U, \chi) \mapsto \omega,$$

is an infinite covering map.

The symmetric group $\mathcal{S}_N$ acts on $\Omega_{\mathcal{L}}^{(N)}$ by deck transformations via

$$(\omega, U, \chi) \cdot \gamma := (\omega, U, \chi \circ \gamma).$$

The the blow-up of $\Omega_{\mathcal{L}} \rtimes \mathbb{R}^d$ by $\Omega_{\mathcal{L}}^{(N)}$ carries a 2-action by $\mathcal{S}_N$.

Restricting the cover $\Omega_{\mathcal{L}}^{(N)} \to \Omega_{\mathcal{L}}$ to

$$\Omega_0^{(N)} := \{(\omega, U, \chi) : U \subset \mathcal{L}_\omega, |U| = N, \quad \chi : \{1, \cdots, N\} \xrightarrow{\sim} U, \quad \chi(1) = 0\}$$

gives a cover $\Omega_0^{(N)} \to \Omega_0$. We denote the blow-up of the groupoid $\mathcal{G}$ by $\Omega_0^{(N)}$ by $\mathcal{G}_N$. The groupoid $\mathcal{G}_N$ also carries a 2-action of $\mathcal{S}_N$, but it does not arise from deck transformations.

Theorem (Mesland-Prodan 2021)

Let $\sigma : \mathcal{S}_N \to \{\pm 1\}$ be the sign representation. The C^*-algebra $M(C^*_\sigma(\mathcal{G}_N, \mathbb{C}))$ contains all Galilean invariant Hamiltonians for the dynamics of N-fermions.

Work in progress: the bi-equivariant groupoid C^*-algebras $C^*_\sigma(\mathcal{G}_N, \mathbb{C})$ and $C^*_\sigma(\Omega_{\mathcal{L}}^{(N)} \rtimes \mathbb{R}^d, \mathbb{C})$ are Morita equivalent in a way compatible with the 2-action of $\mathcal{S}_N$.

- Use point patterns in groups to model configurations of resonators with or without internal structure;
- Single out a family of dynamical matrices or Hamiltonians;
- The transversal groupoid C^*-algebra of a pattern is the smallest C^*-algebra containing the dynamical matrices;
- K-theory and representation theory are governed by the Mortia equivalence class of the pattern groupoid;
- Topological phases and bulk boundary correspondence can be understood in this context;
- Interacting systems of finitely many particles can be interpreted through a blow-up construction.