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Proposal for Research Training Group 2491

Fourier Analysis and Spectral Theory

Second funding period

Speaker: Thomas Schick

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1 Profile of the Research Training Group

Thematic focus. The core of this Research Training Group is modern Fourier analysis and spectral theory. These important classical disciplines in analysis provide powerful tools for many other areas of mathematics. In this RTG, we focus on their use in topology and analytic number theory. The applications include: the investigation of differential and pseudodifferential operators on manifolds with special geometric structure using adapted pseudodifferential calculi; in the spectral and index theory of sub-Laplacians and other hypoelliptic operators arising from geometry; in the context of quantum field theory, where microlocal methods provide powerful tools for renormalisation approaches; in the study of C*-algebras that may serve as observable algebras in quantum field theory; as a vehicle to translate properties of groups into spectral properties of their Cayley graphs, so that Fourier analytic methods apply; new criteria for and applications of Kazhdan's property, expander graphs, and expander complexes; arithmetically enhanced versions of Fourier analysis, like arithmetic harmonic analysis for improvements of the classical circle method.

The history of close and fruitful interactions between Fourier analysis and analytic number theory dates back to the first proof of the prime number theorem, based on analytic properties of the Riemann zeta function. Today, the power of Diophantine analysis is used in modern applications of the circle method, and Fourier analysis is the key toolkit to obtain arithmetic information such as local-to-global principles. Indeed, the investigation of symmetric spaces or more general Riemannian manifolds is one of the most fascinating research topics at the interface of spectral theory, geometry and – if the manifold carries additional arithmetic structure – analytic number theory. This can be summarised in the famous quote "Can you hear the shape of a drum?". On a mathematical level, the relevant questions comprise the study of spectral properties of operators such as the Laplacian and sub-Laplacians (or generalisations thereof) in combination with geometric-topological invariants such as volume, the geometry at the boundary in non-compact cases, or finer properties such as analytic L^2-invariants. Mathematical tools to establish this connection include, for instance, trace formulas and index theory, both of which are featured prominently in several projects in Section 2.

Qualification concept. The RTG has established a vibrant school of early career researchers at the forefront of current research which we hope and expect to continue. We combine the expertise of PIs Bahns, Bauer, Meyer, and Witt on various facets of analysis and the internationally strong group of analytic number theorists in Göttingen (Brüdern, Schindler) with the geometric perspective of Schick, Vigolo, and Zhu. We will continue to guide our doctoral researchers to conduct excellent research and to communicate their research interests and results across the disciplines. The core idea of the qualification programme is to enrich the training and research of our doctoral researchers by providing them with a broad perspective beyond their thesis, which requires deep specialised expertise.

One of the cornerstones of our qualification concept continues to be a special advanced course: the "RTG lecture". We offered such a course during the first funding period and in a previous RTG as well. This course covers a variety of fields that are particularly relevant for the projects described in Section 2. Thus it provides a common knowledge base for all doctoral researchers in the RTG. The "RTG colloquium" and topical summer and winter schools give the doctoral researchers a hands-on experience with current research topics in the field. The doctoral and post-doctoral researchers of the RTG are actively involved in choosing the themes and speakers. This is discussed both during the weekly RTG tea meetings and during dedicated sessions at the RTG retreats. We let our doctoral researchers participate in our extensive international collaborations. If appropriate for the individual situation, we implement an extended stay of several months abroad to broaden their horizons both personally and mathematically.

These core parts of the qualification programme are complemented by a variety of teaching formats with a tradition in Göttingen, in particular, reading and study groups and research seminars. All of these measures provide a unique environment for doctoral researchers. They will acquire a strong scientific background and understanding of important parts of modern mathematics, giving them excellent chances in the job market both inside and outside academia. The work of each doctoral researcher is guided and monitored by an individual thesis committee. The Georg-August University School of Science (GAUSS), the graduate school for mathematics and natural sciences at the University of Göttingen, ensures a structured doctoral study programme in mathematics.

Synergies. The scientific activities of the RTG provide the framework for the mathematical synergies that can emerge in this RTG in the triangle of analysis, analytic number theory, and analytic aspects of topology. While analytic number theory uses Fourier and harmonic analysis as a key input, there is also a considerable flow of ideas in the other direction. For instance, number theoretic methods can be used to explicitly analyse the (non-isometric) action of $\mathrm{Sl}_n(\mathbb{Z})$ on the torus $\mathbb{T}^n$, and all known methods to attack the quantum unique ergodicity conjecture use some arithmetic input. Spectral theory of differential operators helps to analyse resolvent algebras on different domains in $\mathbb{C}^n$ which are potential inputs for the construction and understanding of quantum field theories. Topological arguments are intertwined with spectral arguments in the quest to better understand the dynamics of discrete isometry group actions on compact manifolds and expander complexes associated to them. We have designed the RTG to shed light on such interdisciplinary features and to make our doctoral researchers fluent in the respective theories and languages. All doctoral researchers of the RTG should become familiar with the basic analytic techniques, in particular, spectral theory of (bounded and unbounded) operators on Hilbert spaces, Fourier analysis in abelian and non-abelian situations, microlocal techniques and the role of pseudodifferential calculi, analysis on Lie groups and symmetric spaces, which in one way or another are common to all the projects described in Section 2. Our idea is that the sophisticated specialised methods needed in these specific projects shall be complemented and challenged by the input from adjacent fields.

2 Research Programme

Modern Fourier analysis and spectral theory are used in a variety of contexts. The application of tools from these areas in topology and analytic number theory is the common thread of our research programme, both in the first and the second funding period. A number of important ideas and tools—such as spectral decompositions, Fourier duality, microlocal analysis, and analysis on Riemannian manifolds—are common to our entire research programme.

We start with a short synopsis of the research projects, along with their interrelations, and put them into perspective. Along the way we will, in particular, demonstrate stable edges between the vertices of the triangle of analysis – analytic number theory – topology. Complete details will be given in the forthcoming subsections.

In the second funding period, a central role will be played by the spectral analysis of (slightly) singular operators. These arise from several geometric situations. In particular, subriemannian manifolds such as nilpotent Lie groups and nilmanifolds will be studied in project areas 2.3, 2.4, 2.7, and also 2.6. In 2.4, the focus is on the classical case of sub-Laplacians on functions, which we are studying, e.g., using Fourier restriction methods. In 2.3, the focus is on developing a theory of subelliptic differential complexes and drawing geometric/topological conclusions. In 2.7 and 2.6 the focus is the construction of C*-algebras out of these operators, leading the way in 2.6 to index theoretic applications.

Project area 2.7 studies Fourier analysis in the more abstract form of representation theory. Its starting point is Nelson's Theorem, which describes what representations of a Lie algebra integrate to representations of the corresponding Lie group. In 2.7, Lie algebra representations are replaced by representations of algebras of differential operators or quantum group deformations of compact groups, and positive energy representations of loop groups. In the latter case, the focus is on finding extra algebraic structure on these representations that equips the resulting C*-algebra with a representation of the string Lie 2-group. A representation of the latter relating to positive energy representations is a prerequisite for certain physical applications and also for index theory with the string 2-group. A common feature of the C*-algebras studied in 2.7 is that they may be interpreted as deformation quantisations of certain physical systems. In project area 2.1, from the physical motivation of classification of the elementary fields of a gauge theory on an asymptotically flat spacetime, the representation theory of the infinite dimensional BMS group $\mathbf{BMS}_4 = C^\infty(S^2) \rtimes \mathrm{SL}(2, \mathbb{C})$ (the group of asymptotic symmetries of such a spacetime) is studied, with one of the goals to better understand the associated Weyl algebras.

Project area 2.2 follows another ansatz to describe important physical systems by C*-algebras. Namely, the unbounded self-adjoint operators that are initially given as input are replaced by their resolvents or the unitary one-parameter groups that they generate. In turn, these are then used to generate a C*-algebra. In finite-dimensional linear toy models, these C*-algebras are contained in Toeplitz C*-algebras. We will explore how much remains of this structure in nonlinear and infinite-dimensional models.

The methods we plan to employ in the project areas mentioned so far include a variety of tools from harmonic analysis, such as microlocal constructions of appropriate pseudodifferential calculi, explicit calculations of spectra and heat kernels using generalised Fourier decomposition, and the spectral zeta functions and their analytic properties. Zeta functions and their relatives are also a key tool to encode arithmetic information.

Fourier analysis in arithmetic situations comes up in the large project area 2.5. The power of Fourier analytic techniques in Diophantine analysis is demonstrated in 2.5.3 and 2.5.4, featuring modern variants of the Hardy–Littlewood circle method. More geometric aspects of harmonic analysis provide fine information on the distribution of solutions of Diophantine equations in 2.5.1. It will also be relevant to access arithmetic information through a Fourier transform in 2.6.3, in order to understand the Roe algebra of $\mathrm{Sl}_n(\mathbb{Z})$ acting on the torus. The underlying principle of 2.6 that the geometry at infinity governs many of the relevant features also comes up in 2.5.3, where equivariant partial compactifications are used together with harmonic analysis and the circle method to control the asymptotics of the number of rational points. The problems mentioned above are linked by the fact that Fourier analysis is enhanced by number theory in the guise of lattice point problems and Diophantine considerations. Put differently, Fourier and harmonic analysis are tailored to encode arithmetic phenomena.

Implicit (and explicit) in many of the projects just discussed – such as 2.3, 2.4 – is the question of the connection between geometric properties of spaces acted on by groups and spectral properties of certain invariant operators (such as the Laplacian). This theme features prominently in 2.8, where we ask to what extent one may choose a metric to achieve a determined band-gap structure of the spectrum of this operator. A key tool here is Fourier analysis in the form of Bloch–Floquet theory.

We now turn to the detailed and explicit description of the research programme. The following subsections, ordered alphabetically with respect to the first PIs name, describe specific project areas and list a selection of 27 possible thesis projects. The work on some of these projects has already begun with doctoral researchers of the second cohort, as will be detailed below. All together, the suggested projects provide ample material for the final cohort of 11 doctoral researchers to be hired in the third funding period, and additional associated doctoral researchers to be integrated along the way.

2.1 Microlocal and representation theoretic methods in quantum field theory (Bahns)

All interesting interacting quantum field theories involve terms which are ill-defined, and physicists have developed elaborate tools to "renormalise" them, that is, to assign finite values. In the first funding period, doctoral student **Arne Hofmann** studied the renormalisation problem of QFT in the framework of Lagrangian distributions, that is, distributions which are conormal with respect to a Lagrangian submanifold Λ of the cotangent bundle T^*M. Locally, a Lagrangian distribution is given as an oscillatory integral (ubiquitous also in the analytic theory of automorphic forms). The challenge of the global theory is that it has to be set up in a geometric, coordinate-independent way. Lagrangian distributions provide the natural framework for the renormalisation problem as the (distinguished) fundamental solutions of the partial differential operators in question are one-sided paired Lagrangian distributions [65, 112]. For such distributions, a symbolic calculus is available [93].

Regarding constructive aspects in QFT, a result by Bahns and Rejzner [11] shows that in the framework of perturbative algebraic quantum field theory, the S-matrix of the Sine Gordon model on 2-dimensional Minkowski space (hyperbolic signature) is constructible as a unitary operator. Later, Bahns, Fredenhagen and Rejzner have shown that the Haag–Kastler net of von Neumann algebras of local observables can be constructed explicitly [10] – and hence, the framework indeed provides a completely new approach to constructive QFT. Until now, results on exact models had mostly been restricted to the elliptic signature case, and a subsequent Wick rotation was needed. In his ongoing thesis project, **Fabrizio Zanello** studies the conserved currents of the model. Again, this construction is based on renormalisation techniques, which are tackled in the framework of microlocal analysis [45] .

In the second funding period, the focus will move towards gauge theory on asymptotically flat Lorentzian spacetimes. The physical motivation is the idea that the representations of the symmetry group should classify elementary fields in such spacetimes. The asymptotic symmetry group of a four-dimensional asymptotically flat spacetime is the Bondi–Metzner–Sachs group $\mathbf{BMS}_4 = C^\infty(S^2) \rtimes \mathsf{SL}(2,\mathbb{C})$. **Lennart Janshen**, a doctoral researcher in the second cohort, is working on understanding the irreducible unitary representations of $\mathbf{BMS}_4$, using the Mackey machine of induced representation. Representations of groups like $\mathbf{BMS}_4$ and induction of representations also play an important role in 2.7, where the aim is to construct C*-algebras.

Unlike the more classical Poincaré group, $\mathbf{BMS}_4$ is infinite-dimensional. In addition, its representations depend on the choice of a topology on it. McCarthy [108–111] studied the representations in an L^2-topology, as well as in the nuclear C^∞-topology on $C^\infty(S^2)$. He generalised the Mackey theory of induced representations. A special spin-zero representation associated to the stabiliser group $\Delta \subset \mathsf{SL}(2,\mathbb{C})$ (the double covering of the two dimensional Euclidean group) is interpreted in [58] as a mass-less field theory supported at lightlike infinity.

Janshen aims at understanding the irreducible representations with positive squared mass associated to the stabiliser group $\mathsf{SU}(2) \subset \mathsf{SL}(2,\mathbb{C})$. We hope to construct a related field theory on the asymptotically flat spacetime. Along the way, we hope to construct an injective $*$-homomorphism from the Weyl algebra associated to the conformally invariant Klein–Gordan field theory on the spacetime into the Weyl algebra of the associated field theory, similar to [58]. Such constructions will be related to the study of resolvent algebras of 2.2.

Title of (ongoing) thesis project:

- Asymptotic symmetries in classical gauge theory.

2.2 Toeplitz quantisation, resolvent algebras and symmetric spaces (Bahns, Bauer)

The study of *canonical operators* in quantum mechanics has a long standing history and goes back to the fundamental ideas of Weyl and von Neumann. A mathematically rigorous and classical approach is based on operator theoretical methods and encodes the canonical commutation relation through a C*-algebra: the Weyl or CCR algebra. More precisely, let (X,σ) be a symplectic vector space and consider a real linear map ϕ into a space of essentially self-adjoint operators on a Hilbert space $\mathcal{H}$ with a common dense and invariant core, such that

$$[\phi(f),\phi(g)] = i\sigma(f,g)\mathsf{Id}. \tag{1}$$

The well-known Weyl algebra $\overline{\Delta(X,\sigma)}$ is defined as the unital C*-algebra generated by the unitary operators $\exp(\mathrm{i}\phi(f))$ for $f \in X$. Independently of a concrete representation in $\mathcal{L}(\mathcal{H})$, the Weyl algebra can be characterised more abstractly by the Weyl relations.

More recently, Buchholz and Grundling have proposed the *resolvent algebra* $\mathcal{R}(X,\sigma)$ as an alternative C*-algebraic model with the aim of circumventing some known weaknesses of the Weyl algebra (see [46] for details). We recall that $\mathcal{R}(X,\sigma)$ is a C*-algebra defined by abstract relations. In case of a concrete representation inside $\mathcal{L}(\mathcal{H})$ it can be identified with the unital C*-algebra generated by (certain) resolvents of $\phi(f)$ for $f \in X$. On the one hand, the resolvent algebra allows new physical applications such as the encoding of dynamics in an operator-theoretic model. On the other hand, these algebras and their generalisations described below have interesting algebraic-analytic structures, which even in the setting of a finite-dimensional symplectic space X are not yet fully understood. It remains an important task to analyse resolvent algebras and their generalisations under mathematical aspects and physical relevance.

If X is finite-dimensional, we may consider the model case $X = \mathbb{C}^n \cong \mathbb{R}^{2n}$ equipped with the standard symplectic form σ. Both the Weyl algebra and the resolvent algebra may be studied in their Fock–Bargmann representation, that is, inside the space of bounded operators on the Fock–Bargmann space $\mathcal{H} = \mathcal{F}^2(\mathbb{C}^n)$ of Gaussian square-integrable entire functions on $\mathbb{C}^n$. As was observed in [56], the Weyl algebra $\overline{\Delta(\mathbb{C}^n, \sigma)}$ has a representation as a Toeplitz C*-algebra inside $\mathcal{L}(\mathcal{H})$. More precisely, $\overline{\Delta(\mathbb{C}^n, \sigma)}$ is the C*-algebra generated by the Toeplitz operators with almost periodic symbols. As a consequence, operators in the Weyl algebra have strongly localised kernels and therefore are in various aspects well-behaved (see [173] for details). More recently, it was observed in [19] that an analogous result holds for the resolvent algebra $\mathcal{R}(X, \sigma)$. Based on the shift-invariance of $\mathcal{R}(X, \sigma)$, a convenient tool in the analysis of Toeplitz operators is *Quantum Harmonic Analysis* (QHA) and the notion of *corresponding operator and symbol spaces* as developed by Werner [169] (for the framework of Toeplitz C*-algebras, compare its formulation in [74]).

In the present project, we propose to study C*-algebras generated by resolvents of essentially self-adjoint operators extending the models described above.

The construction and study of C^*-algebras built from explicitly given operators links this project to the study of explicit examples of Roe C*-algebras to be studied in project area 2.6.3, such as $\mathrm{Sl}_n(\mathbb{Z})$ acting on the torus. Even closer is the construction of C*-algebras in project area 2.7. The C*-algebras in 2.7 also provide deformation quantisations for certain physical systems. They tend to have "too few" representations because extra conditions are imposed on the representations of the given unbounded self-adjoint operators in order to get a C*-algebra with exactly the same representation theory. In contrast, the Weyl and resolvent C*-algebras studied in this project area have "too many" representations, namely, they have representations where the original unbounded generators are mapped to ∞. These representations are related to the boundary map in the Toeplitz C*-algebra extension.

The first extension we propose to study replaces $(\mathbb{C}^n, \sigma)$ by an infinite-dimensional separable complex Hilbert space H equipped with a Gaussian measure and a symplectic form σ. Extending the definition of the Fock–Bargmann space, we define $\mathcal{F}^2(H)$ to be the L^2-closure of complex analytic polynomials on H. We propose to study representations of the resolvent algebra $\mathcal{R}(H, \sigma)$ inside the Toeplitz C*-algebra over $\mathcal{F}^2(H)$ (see [19] for some definitions and first results).

We would like to understand whether $\mathcal{R}(H, \sigma)$ and $\overline{\Delta(H, \sigma)}$ are still Toeplitz C*-algebras and whether there are unique corresponding symbol spaces. To our best knowledge, no version of *QHA* or a *correspondence theorem* is presently available in this situation and one may have to extend the theory to the setting of functions in infinitely many variables. We expect to observe new effects due to the lack of a translation-invariant measure and lack of the corresponding L^1-space on H, which are natural ingredients of QHA over $\mathbb{C}^n$. Moreover, the theory of Toeplitz operators and generated C*-algebras is not well-understood in this framework. The peculiarities of infinite-dimensional measure theory and the topological properties of the space of holomorphic functions on H (such as its non-nuclearity as a Fréchet space) require new ideas and a systematic analysis of the quantisation model.

Secondly, we aim to generalise the resolvent algebra from the linear setting of symplectic vector spaces (X, σ) to suitable classes of symplectic manifolds (M, ω). In order to develop appropriate definitions, it will be useful to analyse concrete examples of how the symplectic structure of M can be encoded in the form of a C*-algebra generated by resolvents. We may start from the observation that the canonical operators in the Fock–Bargmann representation appear as generators of strongly continuous unitary one-parameter groups of weighted shift operators on $\mathbb{C}^n$ (Weyl operators). Hence we will first consider symplectic manifolds with complex structure and having a large group of symmetries. Starting with the complex one-dimensional case, we plan to consider first the unit disc $\mathbb{D}$ with the Bergman metric and corresponding symplectic form σ_B. Instead of the Fock–Bargmann space we consider the Bergman space of holomorphic L^2-functions on $\mathbb{D}$. Replacing the shifts on $\mathbb{C}^n$ by the Möbius transforms on $\mathbb{D}$ leads to two families of essentially self-adjoint operators, $\phi(y,z)$ parameterised by the complexified tangent bundle $(y,z) \in T\mathbb{D} \cong \mathbb{D} \times \mathbb{C}$ and $(D_y)_{y \in \mathbb{D}}$ parameterised by $\mathbb{D}$, respectively. These operators fulfil a variant of the canonical relations (1):

$$[\phi(y,z), \phi(y,u)] = i\sigma_B(z,u)D_y. \tag{2}$$

We will analyse the structure of the C*-algebra generated by resolvents of $\phi(y,z)$ and its intersection with the full Toeplitz algebra on the Bergman space. Which conclusions can now be drawn from the invariance of the algebra under symmetries of the domain (that is, conjugation by the unitary maps encoding the automorphisms)? Again, no analogues of QHA and *correspondence theory* are known in this setup. We plan to investigate whether and how one can develop such a theory, or whether there are natural obstructions along the way.

In a second step, we consider the family of standard weighted Bergman spaces on $\mathbb{D}$ with weight parameter $\lambda > -1$. The above construction gives a family of resolvent algebras indexed by λ. We study the deformation of these algebras and possible limit objects as $\lambda \to \infty$.

More generally, we aim to extend the analysis of the resolvent algebra to the case of complex domains of dimension $n > 1$ (see [68, 164] or other quantisation models with symmetries [20]). We may choose $M = \mathbb{B}^n$ to be the open unit ball in complex n-space or, more generally, a Hermitian symmetric manifold of non-compact type. Via the Harish-Chandra embedding, M is realised as a bounded symmetric domain in $\mathbb{C}^n$ equipped with the Bergman metric. As before we consider the weighted and unweighted Bergman spaces over M. generalising the case $M = \mathbb{D}$, we define the notion of a resolvent algebra over (M, σ_B) and we plan to analyse its C*-algebraic properties. We expect new and interesting phenomena caused by the multi-dimensional setup (such as families of commutative subalgebras, which can be studied via Gelfand theory) and the higher complexity of the geometric structure.

We plan the following thesis projects:

- QHA, Toeplitz operators and resolvent algebras in the Fock–Bargmann representation over infinite-dimensional symplectic spaces
- Resolvent algebras for symmetric manifolds: from unit disc model to bounded symmetric domains.

2.3 L^2-invariants and harmonic analysis (Bauer, Meyer, Schick)

The spectral theory of the Laplace-Beltrami operator provides powerful and interesting invariants of manifolds that, despite being defined using a Riemannian metric, are actually of topological nature. Our focus is on L^2-invariants which use regularised (von Neumann) traces on non-compact manifolds and questions derived from them. We aim to continue to investigate Novikov-Shubin invariants and L^2-torsion in geometric situations where tools from harmonic analysis are available.

In the first period, we focused on the explicit study of the Novikov-Shubin invariants of non-linear semi-simple Lie groups—with special emphasis on the universal cover of $SL(2,\mathbb{R})$ for which the full spectrum of the form Laplacian and the Novikov-Shubin invariants are computed in the PhD project of **Zhicheng Han**—and of certain classes of nilpotent Lie groups, with new computations of Novikov-Shubin invariants in the PhD project of **Tim Höpfner**.

In the second funding period, we plan to deepen the analysis of nilpotent Lie groups, exploiting in particular the underlying structure of subriemannian and filtered manifold. We will focus on the spectral theory of the associated hypoelliptic operators and will develop the theory of geometric operators of this type much beyond the model case of the hypoelliptic sub-Laplacian on functions. Finally, we plan to investigate the index theory of hypoelliptic operators in the context of suitable operator calculi.

2.3.1 Sub-Laplace operator on differential forms and applications

A subriemannian manifold is a triple $(M, \mathcal{H}, g)$ with M being a smooth manifold, $\mathcal{H}$ a bracket generating distribution and g a family of inner products on $\mathcal{H}$ smoothly varying with the base point. Under a regularity condition on the subriemannian structure, an intrinsic sub-Laplace operator Δ_{sub} on $C^\infty(M)$ is defined [1]. This operator encodes the subriemannian geometric structure on M and generalises the Laplace-Beltrami operator in Riemannian geometry. Based on the bracket generating property (also known as the *Hörmander condition*) of the distribution, Δ_{sub} is known to be hypoelliptic. The sub-Laplace operator acting on functions as well as the induced heat or wave equation have been studied via analytic or probabilistic methods with applications to the spectral theory of subriemannian manifolds, see [22, 51, 69, 116, 166].

Graded nilpotent Lie groups play the role of a local model for M and carry an induced subriemannian structure themselves. An important example are the Heisenberg groups, which appear as local linear model at each point of a contact manifold. The analysis of graded nilpotent Lie groups G in the framework of subriemannian geometry therefore serves as a first approximation for the general case. Based on the splitting of the corresponding Lie algebra

$$\mathfrak{g} = \bigoplus_{\ell=1}^{r} \mathfrak{g}_\ell \quad \text{where} \quad [\mathfrak{g}_1, \mathfrak{g}_\ell] \subseteq \mathfrak{g}_{\ell+1},$$

one obtains a family δ_λ of non-isotropic dilations on G as an extension of the map $\delta_\lambda(\exp(X_\ell)) = \lambda^\ell(\exp(X_\ell))$ for $X_\ell \in \mathfrak{g}_\ell$ and $\lambda \in \mathbb{R}$. The sub-Laplace operator Δ_{sub} on G is a left-invariant second order differential operator which is homogeneous of order two with respect to δ_λ.

In this project we are planing to study a suitable extension of the sub-Laplace operator from functions to differential p-forms on M. One first observes (see [141, 143]) that the dilation can be extended to the whole differential algebra. However, the de Rham differential d on p-forms $(p \geq 1)$ is not homogeneous with respect to the weights and rather splits into homogeneous parts. As a consequence, it was suggested by Rumin [141] to study suitable filtered complexes based on the dilation properties of d. We expect to further develop the concept of hypoellipticity for complexes and then prove hypoellipticity of such a complex by deriving suitable a priori estimates. Preliminary work in this direction is carried out in [60]. A follow-up goal is then to construct parametrices in an adapted pseudodifferential calculus. Similar questions will then be studied in the case of compact nilmanifolds $\Gamma\backslash G$ where Γ is a lattice in G.

In many concrete cases (including all compact nilmanifolds [69]), it has been shown that the spectral zeta function obtained from the sub-Laplacian acting on functions has a meromorphic extension that is analytic near zero. Then the *regularised determinant* of Δ_{sub} on functions is defined. It is natural to ask whether such results extend to the case of higher order differential forms, in which case one can define and study a subriemannian version of the analytic torsion and L^2-analytic torsion as done in special cases in [3, 81], among others. Here, we will investigate three problem areas: *structural properties (vanishing results, variational formulae); explicit computations for model spaces; relation to the Riemannian analogue*.

The Novikov-Shubin invariants α_p of nilpotent Lie groups (or a Riemannian manifold with cocompact action by a discrete group) can be obtained from the asymptotic behaviour at large times of the Riemannian heat kernel on differential p-forms.

In the second funding period, we will introduce subriemannian analogues of the Novikov-Shubin invariants and will study them under suitable hypoellipticity assumption. A focus will be their geometric significance, in particular their invariance properties in the framework of subriemannian geometry, and also the relation to the classical Riemannian invariants. In fact, in [142] and in case of a graded nilpotent group, Rumin proves Carnot-Carathéodory ellipticity of the de Rham complex. This means that induced Laplacians are maximally hypoelliptic, which is equivalent to the existence of a parametrix and extends the well-known hypoellipticity of the sub-Laplace operator on G in the scalar case.

We propose to study these questions initially for nilpotent Lie groups with additional structure coming from a Clifford module action. In such cases, additional algebraic tools are available and can be applied in the analysis. More precisely, these are the H-type and pseudo-H-type groups introduced by Kaplan and Ciatti [55], respectively.

2.3.2 Hypoelliptic index theory

The analysis that enters the study of hypoelliptic operators may also be used to describe the K-homology class of an invariant differential operator of this type on a graded nilpotent Lie group G. These differential operators are studied via their parametrices using adapted pseudodifferential calculi for filtered manifolds. Such calculi are built and studied in [70], but there is also the approach proposed in [67] to use suitable groupoids. In the first funding period, it was shown that these calculi can be obtained by combining the groupoid approach with Rieffel's construction of generalised fixed point algebras. Moreover, an abstract reduction to the Atiyah-Singer index theorem of the index problem for a Rockland operator was achieved. We want to push this further and explicitly solve index problems on filtered manifolds. The PIs have considerable expertise with this K-theoretic machinery and the groupoid approach to index theory (compare, for instance, [25, 66]). However, the general techniques alone cannot solve the problem because their end result still involves a map that is only defined as the inverse of a certain isomorphism. Namely, the principal symbol belongs to a certain non-commutative C^*-algebra. Its K-theory is isomorphic to that of the unit co-sphere bundle S^*M in the manifold M. It remains to solve the question which K-theory class on S^*M corresponds to a given principal symbol. The Rockland condition, which is the analogue of ellipticity in this context, demands that the symbol is invertible. The problem is to use this Rockland condition to describe the class through finite-dimensional data extracted from the operator, in a way that still works for bundles of graded nilpotent Lie groups. This is a key ingredient in the hypoelliptic index theorem by Baum and van Erp [24]. Recent approaches also apply cyclic homology techniques [77]. This is a promising route were we also have experience [133].

2.3.3 Heat kernel formulae and topological invariants

Above, we described how asymptotic analysis and dilation techniques allowed Rumin to compute the Novikov–Shubin invariants of the Heisenberg groups in all dimensions. Instead of using such asymptotic results, it is interesting to calculate the heat kernel of the form Laplacian on nilpotent Lie groups explicitly. This is a daunting task and is expected to be possible only in very special situations, but if achieved it will provide very detailed and deep information. Indeed, in case of a step-2 nilpotent Lie group the heat kernel of the Laplacian on 0-forms has a concrete integral representation involving hyperbolic functions (the Beals-Gaveau-Greiner formula and its extension, [51]). So far, there are no precise formulae for the heat kernel of the function Laplacian on groups of step three or larger. Hence in this project we are planing to tackle the above problem initially for groups of step two. Preliminary results are already available in [23], where the heat kernel of the Laplacian acting on one-forms in case of the Heisenberg group was calculated in an explicit form as a concrete matrix with operator entries. We will study approaches to generalise such formulae to higher order differential forms and other step-2 nilpotent Lie groups. This might involve an integral decomposition into Grushin type operators as studied in Project Area 2.4. Once established, we will investigate whether and how such formulae serve as a starting point for an asymptotic heat kernel analysis which allows to directly calculate the Novikov-Shubin invariants for certain groups like the Heisenberg group. This should allow to confirm with a different method the results obtained via degeneration analysis [141, 143], and we hope to cover new cases as well. Explicit spectral calculations are also relevant in parts of Project Area 2.6.

A bolder goal is then to apply the explicit spectral analysis in order to calculate L^2-torsion. In a first step we will consider groups in low dimensions and/or with additional structure, such as the above mentioned H-type or pseudo-H-type groups.

The development of the above theory offers many attractive questions for several doctoral researchers. Preliminary titles of thesis projects in this direction could be:

- Analysis of hypoelliptic subriemannian differential complexes
- Subriemannian analytic torsion
- Index theory for hypoelliptic invariant differential operators on (bundles of) graded nilpotent Lie group.
- Explicit heat kernel formulae on nilpotent Lie groups and their geometric applications

2.4 Analysis on nilpotent Lie groups and nilmanifolds (Bauer, Witt)

Fourier restriction theorems play a central role in harmonic analysis and PDE. In particular, they provide a classical tool in the proof of Strichartz estimates for various linear and quasi-linear equations [13, 161]. Generalisations from the Euclidean case to nilpotent Lie groups suggest the use of the non-commutative Fourier transform, which increases the complexity of the analysis. Graded nilpotent groups such as the Heisenberg group naturally appear as model spaces in subriemannian geometry and carry an intrinsic hypoelliptic sub-Laplace operator Δ_{sub}. During the last years, various analytic and spectral theoretical aspects of the Schrödinger, heat and wave equations induced by the sub-Laplacian have been studied [116, 166]. However, even in the case of step-2 groups the analysis is far from being complete. In the setting of the

Heisenberg group, the authors in [13] apply Fourier restriction estimates in order to prove Strichartz estimates for the subriemannian Schrödinger and wave equations. Extensions of these results can be considered for more general nilpotent Lie groups of step two.

In case of H-type groups Strichartz estimates are known in some cases: Bahouri, Gérard, and Xu [17] proved that there are no global dispersive estimates on the Heisenberg group for the Schrödinger equation. This result was generalised by Del Hierro [88] who showed that there are dispersive estimates on H-type groups for the wave equation and also for the Schrödinger equation, the latter if the dimension of the centre p is at least two. The latest result in this direction was by Bahouri, Fermanian-Kammerer, and Gallagher [15] for 2-step stratified groups who, under a maximal-rank condition, established, again for the Schrödinger equation and now for spectrally localised data, a dispersive estimate with a decay rate of $t^{-(p+k-1)/2}$ as $t \to \infty$, where k is the dimension of the radical of the canonical skew-symmetric form. In addition, restriction theorems for the Heisenberg group and then for H-type groups were proven in [13, 104], while local dispersive estimates on the Heisenberg group can be found in [16].

2.4.1 Restriction operators

Starting from a positive self-adjoint differential operator D and based on the spectral theorem, a family of restriction operators can be defined. In the classical case (Stein-Tomas restriction theorem) D is chosen as the standard Laplacian on the Euclidean n-space. In [122] the author considers restriction operators induced by the sub-Laplacian on the Heisenberg group H_n and their mapping properties between mixed L^p spaces. The analysis is based on the more elaborate representation theory of H_n. More recently, Liu and Wang in [104] have studied similar questions for operators induced by sub-Laplacians of H-type groups.

In this project we consider the case of pseudo-H-type Lie groups G, which were introduced by Ciatti [55]. These are step-2 nilpotent Lie groups generalising the class of H-type groups. More precisely, the complement to the centre of the corresponding Lie algebra admits the structure of a Clifford $\mathcal{C}\ell_{r,s}$-module. If G is not an H-type group, then G does not fall into the class of Metivier groups. We expect interesting new effects in the analysis of associated PDEs and restriction operators, originating from the degeneracy of the matrix of structure constants and requiring new methods of investigation. We aim to analyse restriction operators for the sub-Laplacian on G, their mapping properties and some applications.

In many examples, by descending the sub-Laplacian through a submersion (with additional conditions) from a subriemannian manifold M to a base manifold B we obtain a positive sum-of-squares operator $\mathcal{G}$, which we call *Grushin operator*. This construction generalises the classical case where $M = H_3$ is the three dimensional Heisenberg group and $B = H_3/L \cong \mathbb{R}^2$ arises as a quotient of H_3 by a one-dimensional subgroup. In model cases the Grushin operator may be seen as the Laplace operator on B corresponding to a singular Riemannian metric (e.g. Grushin plane, Grushin sphere, or Grushin cylinder) and is of lower complexity due to the reduction of the space dimension. For such model operators $\mathcal{G}$, which are degenerate along a submanifold of B, we will also study restriction operators and their applications to associated PDEs in singular Riemannian geometry.

2.4.2 Dispersive estimates on homogeneous groups

This part of the project aims at understanding the dispersive behaviour of waves on homogeneous groups. It has already started to be taken up by doctoral researcher **Marvin Schmidt** from the second cohort. One of the goals is to establish Strichartz estimates, with possible applications to nonlinear PDEs [84, 85, 145] (concerning global existence, minimal regularity, etc.). Indeed, we aim at generalising the results [13, 15–17, 88, 104] mentioned above from the H-type group case to the case of 2-step stratified groups (aka Carnot groups of step 2). A mathematical tool to achieve this is a pseudodifferential calculus on those groups based on the group Fourier transform. The construction of such a calculus still needs to be finalised. General references are [70, 162]. Yang in his thesis [174] made an interesting suggestion of how to generalise the constructions in [14] for the Heisenberg group to more general groups G. This uses explicit knowledge of unitary irreducible representations of G and of the Plancherel measure on the unitary dual $\widehat{G}$ as well as the Hörmander-Weyl calculus for the growth estimation of the amplitude functions, which are operator-valued function living on $\widehat{G}$. What is still missing is the implementation of the correct degenerate behaviour of the amplitude functions at those points of $\widehat{G}$ which are not seen by the Plancherel measure. Here, a description of the group Fourier transform in terms of the canonical skew-symmetric form and its matrix coefficients [100] might prove to be helpful. Moreover, a list of all low-dimensional nilpotent Lie groups up to dimension 6 is available in [125].

2.4.3 Spectral asymptotics of the sub-Laplacian on nilmanifolds

A third project concerns the spectral theory of the sub-Laplacian on compact quotients of pseudo-H-type groups G by co-compact subgroups $\Gamma \subset G$. The left-coset space $M = \Gamma\backslash G$ is called a compact pseudo H-type nilmanifold and we refer to Γ as a lattice in G. The left-invariant sub-Laplacian descends from G to the sub-Laplacian Δ_{sub}^M on M. Based on subelliptic estimates it is known that Δ_{sub}^M has discrete spectrum consisting of eigenvalues with finite multiplicity. In recent years there have been advances in the spectral analysis of compact nilmanifolds. Explicit heat trace formulae have been obtained and, as an application, finite families of Δ_{sub}^M-isospectral, but non-homeomorphic pseudo-H type nilmanifolds were detected, [21, 69].

We aim to further study the inverse spectral problem on M by analysing the spectral zeta and eigenvalue counting function of the sub-Laplacian. A concrete question asks whether one can read the topological dimension of M from the spectrum of Δ_{sub}^M. As in the classical case of the Laplacian on a Riemannian manifold the subriemannian analogue of the heat equation and the wave propagator will be crucial. In [159] the author has determined the asymptotic behaviour of the eigenvalue counting function in the special case of Heisenberg manifolds from the explicitly known spectrum of Δ_{sub}^M. Whereas the leading term in the asymptotic is known in general, [159] interestingly observed that the error terms are better than for the standard Laplacian on a torus.

The eigenvalue counting function shall be studied more generally for pseudo-H-type nilmanifolds by abstract methods from harmonic and microlocal analysis. We hope to gain a better understanding of bounds on the error term and possibly link Strichartz' observation to the structure of the spectral zeta function and its distribution of singularities. Further spectral asymptotics appears by considering a tower of nilmanifolds Γ_n/G induced by a decreasing tower of subgroups $\Gamma_1 \supset \Gamma_2 \supset \cdots$ with trivial intersection. We are planing to investigate the asymptotic behaviour of the spectra of the sub-Laplacians on Γ_n/G as $n \to \infty$, which should be related to the sub-Laplacian on G itself. This analysis may link our project to the study of L^2-invariants in Project Area 2.3.

Topics for prospective/ongoing thesis projects:

- Establish Fourier restriction theorems for the sub-Laplacian on pseudo-H-type groups and applications.
- Further develop pseudodifferential calculi on homogeneous groups based on the group Fourier transform.
- Establish dispersive estimates, Fourier restrictions theorems, etc. and derive corresponding Strichartz estimates (this is already ongoing as project of Marvin Schmidt).
- Derive spectral asymptotics of the sub-Laplacian on step-2 compact nilmanifolds.

2.5 Fourier analysis and Diophantine Equations (Brüdern, Schindler)

The central objects in this part of the proposal are Diophantine problems that may be treated with Fourier analytic methods. There are two lines of attack. On the one hand, Fourier analysis is brought to bear on Diophantine problems. Here Fourier analysis is in the role of a catalytic toolkit, and it yields arithmetic information such as local to global principles for classes of equations, or finer information on the distribution of the solutions. On the other hand, we also propose to refine the current Fourier analytic toolkit itself. The major instrument here is the circle method in its classical and more recent versions. The goal then is to identify new classes of problems accessible to Fourier analytic methods.

Doctoral researchers will require a very good understanding of techniques from analytic number theory, and in particular of the circle method and its variants. Some of the problems below are related to geometric ideas, which are guiding the projects from a technical point of view as well as enable us to put the results into a more general framework. For others, a more substantial background in harmonic analysis is helpful.

2.5.1 Dimension growth

Given a projective variety X over a number field k equipped with a height function H on its k-rational points, the counting function

$$N(B) := \sharp\{x \in X(k) : H(x) \leq B\}, \qquad B \in \mathbb{R}_{>0}$$

is a central object of study, if one wants to understand the arithmetic of X. If X is a Fano variety, then we have very precise expectations for $N(B)$ given by versions of Manin's conjecture. If, however, we aim to assume as little as possible about X, a natural question is to obtain upper bounds for $N(B)$. This is closely related to the dimension growth conjecture, which has seen steady progress in the past few decades mainly through techniques such as the determinant method. In recent work of Huang [91] a very new approach has been introduced. Instead of using the arithmetic information from the variety X, one basically forgets all Diophantine structure and instead considers more generally manifolds in $\mathbb{R}^n$. Using methods purely from harmonic analysis, Huang managed to establish the weak dimension growth conjecture for hypersurfaces in projective space with nowhere vanishing Gaussian curvature. Building on this breakthrough, Schindler and Yamagishi [151] managed to adapt the methods to manifolds of higher codimension under certain curvature conditions. A very interesting feature in [151] is that for higher codimension one obtains in some cases even stronger results than had been previously predicted by the (analogue) of the dimension growth conjecture. In the first part of this RTG project, **Florian Munkelt** [120] has already successfully worked on relaxing some of the curvature conditions. The new approach using harmonic analysis opens a number of doors which we further aim to explore in the second funding period. These include the following questions:

1. *count the number of rational points on submanifolds of $\mathbb{R}^n$ which are at the same time constrained to lie on some algebraic variety, such as for example a quadric,*
2. *use these techniques to establish analogous results for affine problems,*
3. *allow for differing denominators of the rational points, with the goal of understanding analogues of the dimension growth conjecture in more general toric varieties,*
4. *work on number field versions and compare the results with the ones obtained in the higher codimension situation via taking Weil restrictions.*

2.5.2 The distribution of rational points

In another direction we aim to study situations where we restrict to certain classes of projective varieties and aim to establish the Hasse principle as well as asymptotics for the counting function $N(B)$. If one considers a smooth complete intersection in $\mathbb{P}^n_{\mathbb{Q}}$ of sufficiently large dimension depending on its degree, then work of Birch [28] and Browning and Heath-Brown [43] establishes the local-global principle. These results do not have a particularly natural or good dependence on the codimension. In his breakthrough work [144] Myerson managed to get a linear dependence of the number of variables in terms of the codimension. For example, his work [144] shows that the Hasse principle holds for smooth complete intersections of R quadratic forms in n variables as soon as $n \geq 9R$. This is a big step ahead but there are still certain systems of quadrics for which we have already stronger results available. For example consider a single non-degenerate quadratic form over a number field of degree R in at least 5 variables. Then number field versions of the delta method [86] can be used to establish the Hasse principle as well as asymptotics for the number of rational points of bounded height. This is equivalent to studying the arithmetic of its Weil restriction over the field of rational numbers and in particular establishes here the Hasse principle for the corresponding system of R quadratic forms in at least $5R$ variables. One can now ask if there is a way to extend these ideas to other systems of quadratic forms (those coming from a Weil restriction forming a small Zariski closed subset in the naive moduli space). In recent work [44], Browning, Pierce and Schindler have taken up this idea in introducing a concept of generalised quadratic forms over number fields. Moreover number field applications of the delta method are then used to enlarge the class of systems of quadratic forms for which we can establish the Hasse principle. This idea of transferring systems of forms of same degree to number fields objects, is a very flexible one. As a second line of research we propose for this RTG project the following project:

Use the classical circle method to establish the Hasse principle for certain systems of equations of same degree using generalised forms over number fields.

As a third line of research we aim to specialise even further to families of Fano varieties where one already is able to establish the Hasse principle as well as understand the density of rational points. Consider for example a hypersurface $X \subset \mathbb{P}^n_{\mathbb{Q}}$ and a height function H on its $\mathbb{Q}$-rational points. Then a first indication for the distribution of its rational points is the counting function $N(B)$ as above. Though, this does not tell us much about the local distribution of points. For example, one could fix a real point ξ on X and ask for the number of $\mathbb{Q}$-rational points that are of bounded height and close to the point ξ. Peyre in [131] has suggested a research program in order to understand this local distribution, see for example also work of Sardari [147] who managed to get nearly sharp answers for some analogous situations of affine quadrics. In work in progress, the PI Schindler together with Zhizhong Huang, Miriam Kaesberg, and Alec Shute is currently exploring those questions in the setting of projective quadrics. One may also ask about other statistics, as for example motivated in work of Bourgain, Rudnick and Sarnak [40, 41], who study those questions for lattice points on the sphere. For example one may ask for point pair statistics and asymptotically evaluate the sum

$$S_1(B) := \sum_{\substack{x,y\in X(\mathbb{Q})\\ H(x),H(y)\leq B, x\neq y\\ \mathrm{dist}(x,y)\leq r}} 1.$$

for a certain distance function $\mathrm{dist}(x, y)$ and two parameters $r, B > 0$. Or consider a statistic related to the electrostatic energy of points of height bounded by B, given by

$$S_2(B) := \sum_{\substack{x,y\in X(\mathbb{Q})\\ H(x),H(y)\leq B, x\neq y}} \frac{1}{\mathrm{dist}(x, y)}.$$

This third line of research has the following aim:

Consider a smooth hypersurface of large dimension and low degree and use Fourier analytic methods to study fine scale distributions of rational points, such as the local distribution of points and point pair statistics.

The projects above are in the intersection of the research interests of PIs Schindler and Brüdern, and we expect fruitful interactions between these research groups as well as possibilities for co-supervised projects. Possible thesis topics in this project are:

- establishing the Hasse principle for systems of equations of same degree using generalised forms,
- local distribution of rational points and point pair statistics.

The doctoral researcher **Mieke Wessel** from the second cohort has already started with an ongoing project in the direction of the last question.

2.5.3 Integral points on log Fano varieties

The projects in Section 2.5.2 have the goal to deepen our understanding of the arithmetic of projective varieties. In the setting of affine varieties, the natural analogue is to study the set of integral points. This sub-project has the goal of understanding the density of integral points on certain log Fano varieties. More precisely, consider a smooth projective variety X over a number field k with a strict normal crossing divisor D such that the associated log-anticanonical bundle is ample. Then consider a flat integral model $\mathcal{U}$ of $X \setminus D$ over the ring of integers $\mathcal{O}_K$ of K and study the counting function

$$R(B) = \sharp\{x \in \mathcal{U}(\mathcal{O}_K), H(x) \leq B\}, \quad B \in \mathbb{R}_{>0},$$

for a suitable height function H. If one allows the removal of accumulating subsets, there are predictions for the growth of this counting function by Chambert-Loir and Tschinkel [52] providing an analogue to Manin's conjecture in the setting of rational points. Similarly as in the setting of rational points, a large variety of techniques has been used to establish asymptotics for $R(B)$ for certain cases, including the circle method and harmonic analysis methods in the case of partial equivariant compactifications. Recently, Wilsch [171] has for the first time applied the universal torsor method to a certain log Fano threefold, given by the blow-up of $\mathbb{P}^3$ along a smooth conic with a rational point, with different choices for D. After using the universal torsor method, Wilsch reduces to a counting problem on a Diophantine equation, which he can solve by elementary methods in summing over each variable one after another. This is a strategy which works beautifully for his example and leads to a great result in his case, but easily breaks down if one looks at slightly different situations. For example if one considers the problem of the blow-up of $\mathbb{P}^3$ along a smooth conic *without* a rational point, then it is not clear how his method could generalise. The goal of this sub-project is to combine the use of a torsor approach with the circle method for the actual counting problem. There is one additional problem to be overcome, namely, the height function typically leads to a region of counting to which the circle method cannot directly be applied. For this, Blomer and Brüdern [33] have introduced the concept of a flexible form of hyperbola method, which has already been used a number of times in the setting of rational points, see for example [32, 117, 150]. Their method has been further generalised by Pieropan and Schindler [134] to allow for even more general height functions, which is exactly what will be needed in this project. A concrete goal for the ongoing project of doctoral researcher **Anouk Greven** of the second cohort in this project is the following:

Apply a strategy consisting of a torsor approach, the circle method and the hyperbola method to establish an asymptotic formula for the counting function $R(B)$ in new cases of log Fano varieties.

The project starts with the case of a blow-up of $\mathbb{P}^3$ along a smooth conic without a rational point with suitable divisors D to familiarise themselves with the techniques and then later move on to exploring to which other cases these techniques can be applied, guided by the classification of Fano threefolds in [92, 118].

2.5.4 Fourier coefficients of arithmetic origin

This sub-project develops ideas of PI Brüdern proposed for the first funding period under the heading Arithmetic Fourier Analysis. The motivation was to develop a linear algebra for d-th powers. Here, one is interested in the distribution of integral solutions of the homogeneous system

$$a_{i1}x_1^d + a_{i2}x_2^d + \ldots + a_{is}x_s^d = 0 \quad (1 \leq i \leq r) \tag{3}$$

with integer coefficients a_{ij}. The ongoing thesis project of **Tammo Dede** of the second cohort goes along this way already with $d = 1$, using techniques from the geometry of numbers and Hasse principles inspired by [42]. Beyond this, since the equations arise from a linear system, forcing the solutions to be d-th powers, there is a simple but rich structure underneath that also shows on the dual, Fourier analytic side. Brüdern and Wooley [BW07, BW16] dealt with the cubic case $d = 3$ for $s > 6r$, thus touching the limit of what linear Fourier analysis can achieve. In a more experimental direction, these authors [BW15,BW18] noticed that the truth of the Hasse principle for the system (3) can be made to depend on moments of Fourier coefficients "of arithmetic origin". One starts with a suitable, fairly dense set $A \subset \mathbb{Z}$ and sets

$$f(\alpha) = \sum_{\substack{|x| \leq B \\ x \in A}} \mathrm{e}^{2\pi \mathrm{i} \alpha x^d}.$$

For positive real t, and integral h one considers

$$\psi(h) = \int_0^1 |f(\alpha)|^t \mathrm{e}^{-2\pi\mathrm{i}\alpha h}\,\mathrm{d}\alpha.$$

If t is an even natural number then, by orthogonality, $\psi(h)$ is the number of solutions of

$$\sum_{j=1}^{t} (-1)^j x_j^d = n$$

with $x_j \in A$, $|x_j| \le B$. For other values of t no such interpretation seems possible, but the Fourier coefficients $\psi(h)$ still remember their arithmetic origin encoded on f. In [BW15,BW18] the moments

$$\sum_h |\psi(h)|^\nu \tag{4}$$

with $\nu = 3$ and 4 and "exotic" values of t have been explored to obtain the Hasse principle for the equations (3) with $d = 4$, $r = 2$ and 3 and surprisingly small values of s. In these applications one has $\nu = r + 1$ but other relations are possible. In a very natural way, one may view the moments sum (4) as an object dual to a Diophantine system with $\nu - 1$ equations and νt variables, and we are no longer required to have ν or t integral. This points towards a complex interplay between systems (3) as r and s vary. Very recently this has been explored in [BW22a, BW22b,BW23], and it is now also possible to combine the combinatorial methods of [BW16] with the moment method of [BW15] (see [BW23]). A brave doctoral researcher may try to develop a systematic theory to obtain new instances for the Hasse principle for small degree and rather small dimension. This should then incorporate recent progress on moment estimates [BWta1, BWta2]. A more down-to-earth project would be an analysis of the "rounded" analogue, in the sense of Mazur [107], of the system (3). This should be a fairly accessible project. Since this is a largely unexplored set of ideas, there are many possibilities for further thesis projects that concern additive problems with powers. New to this circle of ideas is **Tanja Küfner** of the second cohort, with a project concerning the additive theory of powers that aims to establish an effective version of Freiman's theorem.

2.6 Large scale geometry and harmonic analysis (Meyer, Schick, Vigolo)

"Large scale geometry" is the paradigm of studying non-compact metric spaces from "far away", neglecting all the local information and just focusing on the features of the space at infinity.

This is a powerful paradigm for two essential reasons:

1. there are cases where the structure of an object of study is not sufficient to fully determine a geometry, but it does suffice to define a large scale geometry. Much of geometric group theory builds on this principle: there one wishes to assign to a finitely generated discrete group its Cayley graph as a geometric object, but this assignment depends on the choice of a finite generating set. On the other hand, the large scale geometry of the Cayley graph does not depend on it, and it is hence a well-defined invariant of the group.

2. In other circumstances, the study of the large scale geometry allows to focus on the most relevant features. These then become manageable and computable, while studying the full space would be too unwieldy and overwhelming. This strategy has been implemented successfully in large scale index theory, for instance, in Roe's partitioned manifold index theorem.

Large scale geometry has been developed with its own tools and approaches. One such tool is the use of certain C^*-algebras – in particular Roe algebras – and their K-theory. Along the way come index theoretic and spectral properties in the form of large scale index theory and its application to spectral properties, in particular, to invertibility and non-invertibility of geometric operators. In this way, the study of large scale geometry has strong connections to questions of spectral engineering as discussed in Project Area 2.8, where a large scale index can provide obstructions to the construction of geometric operators (Laplacians, Dirac operators) with predetermined spectral properties.

Very important are also constructive methods, among them cones and warped cones. These are used to form bridges between large scale geometry and the usual small scale geometry of compact manifolds, foliations on them, and the dynamics of group actions. Specifically, to define a (dynamical) warped cone one starts with a discrete group Γ generated by a finite generating set S and acting by isometries on a compact Riemannian manifold M. To this action one associates a large scale geometry $\mathcal{O}_\Gamma(M)$ defined as the disjoint union $(M \times \{k\})_{k\in\mathbb{N}}$ of copies of M, but equipped with a warped scaled metric d_k: by definition, d_k is the largest metric such that $d(x,y) \leq k d_k(x,y)$ and $d_k(x,sx) \leq 1$ for all generators $s \in S$. We also demand that (x,k) and $(x,k+1)$ have distance ≤ 1 for each $x \in M$ and $k \in \mathbb{N}$. This construction (and its foliated counterpart) play an important role in the sequel.

2.6.1 High dimensional expansion

One of the research foci will be on the phenomenon of "high-dimensional expansion" and "higher Kazhdan properties".

Loosely speaking, expander graphs are sequences of graphs of increasing size which are very highly connected. They can be spectrally defined as a sequence of finite connected graphs (X_n) of bounded vertex degree such that the spectra of the graph Laplacians for the graphs X_n have a uniform gap at zero. It is quite hard to concretely construct expander graphs. Most of the known explicit constructions have a group theoretic or arithmetic origin.

One specific way to obtain interesting expanders is through the construction of warped cones. One starting point of our investigations is the following result [168]: (the level sets of) a warped cone are coarsely equivalent to a sequence of expander graphs if and only if the Laplacian on the Cayley graph of Γ twisted by the unitary action on $L^2(M)$ has a spectral gap. Importantly, this holds automatically if Γ has Kazhdan's property (T). This and related work have given rise to a host of expanders with interesting properties. One such property is that they are counterexamples to the coarse Baum–Connes conjecture [64, 149], which we will discuss below.

In recent years, one focus of interest shifted to high-dimensional variants of expanders. It turns out that there are different notions of high-dimensional expansion, generalising the different aspects of the classical concept. A sequence of k-dimensional spectral expanders is then a sequence (X_n) of k-dimensional simplicial complexes (finite and connected and of uniformly bounded vertex degree) and such that in all degrees $0 \leq j < k$ the cohomology vanishes and the combinatorial Laplacians have a uniform spectral gap around zero. A weakening of this notion allows for cohomology groups of bounded dimension.

As already mentioned, the group theoretic property which is fundamental to construct expander graphs is Kazhdan's property (T). One of several definitions says that a discrete group Γ has property (T) if the group cohomology with coefficients twisted by any unitary representation with no non-zero invariant vectors vanishes in degree 0 and 1. As suggested by Bader and Sauer, its k-dimensional strengthening, higher Kazhdan's property (T_k), is then defined by requiring the same up to degree k (related notions have been explored in [7] and [54]). In the previous funding period, we had in mind to address property (T) for Aut(F_4), a problem which has been solved in the meantime outside of the RTG by PI Schick's former doctoral student Nitsche [124].

The specific goal of our research programme is to relate higher expansion properties of the warped cone to higher spectral gap properties of the action of Γ on M. Here, in joint work with de la Cruz Mengual, Vigolo has almost proved that the warped cone $\mathcal{O}_\Gamma(M)$ is a k-dimensional expander if and only if the degree p combinatorial Laplacian computing the cohomology of Γ with coefficients in an appropriate Hilbert completion of $\Omega^q(M)$ has a spectral gap for all $p + q < k$. In particular, if the group Γ has higher Kazhdan property (T_k) then warped cones arising from isometric Γ-actions would automatically be k-dimensional expanders. This would generalise the construction of higher expanders via groups with higher Kazhdan property (T) of [115]. Examples can be constructed using a recently announced result of Bader and Sauer, showing that if Γ is a lattice in a simple Lie group of rank $(k + 1)$, then Γ has property (T_k).

The above result requires that the action of Γ on M is isometric to obtain a unitary representation on $\Omega^*(M)$. One specific task of our project is to go beyond this case. A tantalising example is the standard action of $\mathsf{SL}_n(\mathbb{Z})$ on $\mathbb{T}^n$. We conjecture that its warped cone is an $(n - 1)$-dimensional expander. Here we expect that explicit computations using Fourier analysis in the spirit of [75] will provide the tools to attack this specific problem.

In a different direction, we suggest to investigate results of Künneth type for higher spectral gaps. A goal is then to construct continua of non-coarsely equivalent higher expanders. For classical expanders, this is achieved in [71] using related techniques. Note that for such results it is crucial to go beyond higher Kazhdan groups; in the specific case, by taking the product of a higher rank lattice with a free group. This is because the known examples of groups with higher property (T) are too rigid to give rise to a continuum of inequivalent isometric actions.

2.6.2 Laplacians and sub-Laplacians on foliations

Given a foliated compact Riemannian manifold $(M, \mathcal{F})$, the foliated warped cone construction $\mathcal{O}_\mathcal{F} M$ is analogous to the dynamical warped cone, where the constraint that $d_k(x, sx) \leq 1$ is replaced by the requirement that the leaves of the foliation are not stretched. Equivalently, the foliated warped metric d_k can also be described as the Riemannian metric $d_\mathcal{F} + k^2 d_\mathcal{N}$ obtained by stretching the directions orthogonal to the leaves. The analogy between the foliated and dynamical warped cone becomes evident when Γ is the fundamental group of a compact aspherical manifold N: then the dynamical warped cone $\mathcal{O}_\Gamma M$ is large-scale equivalent to the foliated warped cone $\mathcal{O}_\mathcal{F}(M \times_\Gamma \widetilde{N})$, where $M \times_\Gamma \widetilde{N}$ denotes the quotient by the diagonal action and the foliation is induced by the product structure and has leaves homeomorphic to $\widetilde{N}$.

The goal is now to study the spectral properties of the Laplacians on higher degree differential forms of foliated warped cones. More specifically, we aim to understand situations where these Laplacians have uniform spectral gap. This investigation is very much related to the problem of constructing higher dimensional expanders, as there are discretisation techniques that can be used to move from a smooth to a simplicial setting and vice versa. However, this geometric setting has it own idiosyncrasies. For one, studying higher degree Laplacians on a Riemannian manifold is an interesting and delicate problem in its own right. For another, these questions also connect with the study of hypoelliptic operators as in sub-projects 2.3 and 2.4.

2.6.3 Roe algebras and Baum–Connes

As already mentioned, one of the applications of expander graphs is that they provide counterexamples to the coarse Baum–Connes conjecture [64, 89]. This conjecture says that for a large scale geometry like $\mathcal{O}_\Gamma(M)$ an index map $\alpha\colon K_*^{\mathsf{lf}}(E\mathcal{O}_\Gamma(M)) \to K_*(C^*\mathcal{O}_\Gamma(M))$ is an isomorphism. On the left hand side, $E\mathcal{O}_\Gamma(M)$ is a uniformly contractible topological space which is coarsely equivalent to M (that is, it has uniformly no interesting local topology) and K_*^{lf} is the locally finite K-homology. On the right hand side features the Roe C^*-algebra of the large scale geometric space: a certain C^*-algebra of bounded operators on $L^2(\mathcal{O}_\Gamma(M))$ satisfying local compactness and propagation conditions.

The expander property now allows in suitable situations to construct "ghost projections": projection operators which belong to the Roe algebra and represent non-trivial elements in its K-theory, but which are too non-local to lie in the image of the coarse assembly map α.

All the counterexamples to the coarse Baum-Connes conjecture obtained so far show that the surjectivity of the coarse assembly map is violated. This leaves open the intriguing question whether injectivity of the assembly map may also fail. We expect this is the case. As an approach to proving it, we propose to construct ghost relations instead of ghost projections. Ghost relations are unitaries which conjugate two otherwise distinct projections and that can only be defined in the Roe algebra because of (higher) expansion properties of the spaces. Here, explicit examples might be studied using harmonic analysis. First results which use higher Kazhdan properties to obtain new results about the coarse Baum–Connes conjecture are given in [101].

A very much related problem is to investigate Roe algebras and their K-theory for special examples of warped cones. Two important such examples are the natural action of $\mathsf{SL}_n(\mathbb{Z})$ on the n-dimensional torus and the action of a free subgroup of $\mathsf{SU}(2)$ with two generators by left multiplication on $\mathsf{SU}(2)$. The warped cones given by these examples are of great interest, for instance, because they naturally contain expander graphs [168]. At the same time, they are concrete enough that an explicit study of their Roe algebra seems within reach. The explicit construction and study of resolvent C*-algebras from properties of the underlying operators in project area 2.2 offers related questions. One special feature of the $\mathsf{SL}_n(\mathbb{Z})$-action on the torus is that it interacts particularly nicely with techniques of Fourier analysis. This involves to understand the spectral theory of geometric operators in the picture, which also come up in Project Area 2.3. The advantage of the $\mathsf{SU}(2)$-action on itself is that it is a free isometric action and it is more suited to using tools of representation theory. Explicit K-theoretic computations would greatly help to illuminate the nature of (the failure of) the coarse Baum–Connes conjecture. A PhD project in this area has, focusing on the warped cone for the action of the free group on $SU(2)$ by left multiplication and the K-theory of its Roe algebra has just been started by **Christos Kitsios** as one of the doctoral researchers of the second cohort.

The development of the above theory offers many attractive questions for several doctoral researchers. Preliminary titles of thesis projects in this direction are:

- Higher expansion of the warped cone of $\mathsf{SL}_n(\mathbb{Z})$ acting on $\mathbb{T}^n$
- Higher expanders and Künneth theorems
- Ghost relations in the Roe algebra of higher expanders

- Computations of K-theories of warped cones with applications to the coarse Baum–Connes conjecture.

2.7 Representations of Lie algebroids and higher Lie groups (Meyer, Zhu)

For various algebraic objects, it is interesting to look at their representations on Hilbert spaces. These may be studied by finding a C*-algebra that has naturally equivalent Hilbert space representations. This C*-algebra may be used to study the original representation theory. In some contexts such as deformation quantisation, finding a C*-algebra with suitable properties is the actual goal, and the category of Hilbert space representations is a tool to define it. This project studies several problems of this kind. The resulting C*-algebras are deformation quantisations of more geometric classical systems, and our approach is to pin them down by specifying their representation theory. The C*-algebras studied in project area 2.2 serve the same purpose. They are, however, defined rather differently, by replacing unbounded operators by their resolvents or unitaries and taking the C*-algebra generated by the latter.

The work in this project area during the first funding period concentrated on C*-algebras associated to Lie algebroids and quantum group deformations of compact Lie groups. There remain interesting open questions in this direction, so that they are still discussed below. In addition, we added a third direction that focuses on C*-algebras associated to positive energy representations of certain loop groups, which should carry extra structure that makes them a representation of the string Lie 2-group.

To extract a unique C*-algebra from its representations, we actually need a bit more than Hilbert space representations, namely, representations on Hilbert modules. This is made precise in the concept of a C*-hull for a given class of representations of a *-algebra on Hilbert modules, which is developed by the PI Meyer in [113]. Here, however, we will neglect the need to take into account representations on Hilbert modules to simplify the exposition. A prototypical result for the concept of a C*-hull is Nelson's Theorem. It compares representations of a simply connected Lie group G with representations of its Lie algebra $\mathfrak{g}$. Any representation of G may be differentiated to a representation of $\mathfrak{g}$ by unbounded operators, which is defined on the subspace of smooth vectors. A Lie algebra representation of $\mathfrak{g}$, however, need not come from a representation of G. A necessary and sufficient condition for this is that a certain Laplacian element in the universal enveloping algebra of $\mathfrak{g}$ acts by an essentially self-adjoint operator. We call a representation of $\mathfrak{g}$ integrable if this is the case. And the group C*-algebra of G turns out to be a C*-hull for the integrable representations of $\mathfrak{g}$.

A recurring theme in all examples of C*-hulls studied in this project is that the integrable representations are characterised as those in which a certain Laplacian-like element of a *-algebra acts by an essentially self-adjoint operator. This element satisfies a form of ellipticity in that its smooth vectors are already smooth vectors for the whole representation.

2.7.1 Lie algebroids

One goal in the first funding period was a variant of Nelson's Theorem for Lie groupoids and Lie algebroids. An important special case is the *-algebra Diff(M) of differential operators on a smooth manifold M. This is the enveloping algebra of the tangent Lie algebroid of M. The corresponding groupoid is the fundamental groupoid $\Pi_1(M)$ of M. The doctoral researcher **Geoffrey-Desmond Busche** is about to prove that a representation of Diff(M) comes from a representation of $\Pi_1(M)$ if and only if a certain Laplace-like element in Diff(M) acts by an essentially self-adjoint operator. His thesis should be finished in the first half of 2023.

More generally, let G be a Lie groupoid with simply connected source fibres and let $A(G)$ be its Lie algebroid. Let $\mathsf{Diff}_G(G)$ be the algebra of left-invariant differential operators on G. This specialises to $U(\mathfrak{g})$ if G is a Lie group and to the algebra $\mathsf{Diff}(M)$ if $G = \Pi_1(M)$. There is an element $L \in \mathsf{Diff}_G(G)$ of order 2 that is elliptic along the range fibres of G. We conjecture that $\mathrm{C}^*(G)$ is a C^*-hull for the class of representations of $\mathsf{Diff}_G(G)$ in which L acts by an essentially self-adjoint operator. Roughly speaking, this says that a representation of $A(G)$ on a Hilbert space integrates to a representation of $\mathrm{C}^*(G)$ if and only if L acts by an essentially self-adjoint operator.

While the thesis work of Busche contains many results that will help to prove this more general conjecture, some technical issues will probably remain. One of them is that Nelson's Theorem works with analytic vectors, while bump functions on smooth manifolds tend to produce smooth and not analytic vectors. A second is to understand which "Laplacians" L may be used in the theorem. The more flexible techniques in [138] should help to get around these difficulties. Right now, it seems more appropriate for the PIs Meyer and Zhu to work on these issues, to provide a basis for a doctoral researcher during the third funding period to work on the more general case of a Lie algebroid that fails to integrate to a Lie groupoid. Then the integrating object is a Lie 2-groupoid. This so called Weinstein groupoid was built by PI Zhu in [163]. It seems plausible that a representation of the Lie algebroid integrates to a representation of this Lie 2-groupoid if and only if a suitable Laplacian acts by a self-adjoint operator. To get a C*-hull from this, we first have to define the C*-algebra of a Weinstein groupoid. This goes slightly beyond the existing constructions of C*-algebras from crossed modules of groupoids in [49]. Then the representations of this C*-algebra must be described by an analogue of Renault's Disintegration Theorem for representations of locally compact groupoids. It seems plausible that the integration and disintegration of Lie algebroid representations will work similarly to the case of integrable Lie algebroids. A positive answer to this conjecture would give geometric quantisations for those situations where the integrability of the relevant Lie groupoid obstructs the construction by Hawkins [83].

The Weinstein groupoids mentioned above are among the simplest examples of higher Lie groupoids. It is still straightforward to "differentiate" them, and this gives an ordinary Lie algebroid and not yet a higher Lie algebroid. In general, the integration and differentiation processes between higher Lie groupoids and higher Lie algebroids are still active areas of research. The differentiation of a higher Lie groupoid should produce its tangent complex, a replacement for the tangent space of a Lie group, equipped with suitable extra structure that describes its higher Lie algebroid structure. A construction by Severa for doing this may continue to work for simplicial manifolds that lack the extra conditions that make them higher Lie groupoids. A typical example of such a simplicial manifold is the nerve of a neighbourhood of the unit element in a Lie group, equipped with its partially defined multiplication. This extra generality is relevant because the integration of a higher Lie algebroid often proceeds in two steps, where the first one produces only such a simplicial manifold. The ongoing thesis project of **Florian Dorsch** of the second cohort examines Severa's differentiate method in greater generality. The idea is that "local" versions of the Kan conditions that are required for higher Lie groupoids suffice to perform the construction.

2.7.2 Nelson's Theorem for compact groups and quantum groups

Compact Lie groups such as $\mathsf{SU}(n)$ may be deformed to compact quantum groups $\mathsf{SU}_q(n)$. These compact quantum groups may be described by a C*-algebraic compact quantum group due to Woronowicz, which deforms the group C*-algebra of the underlying Lie group, or by a Hopf *-algebra due to Drinfeld–Jimbo, which deforms the universal enveloping algebra of the corresponding Lie algebra. Does Nelson's Theorem have an analogue for these deformed *-algebras? This should characterise when a representation of the Hopf *-algebra integrates to a representation of the quantum group C*-algebra. Instead of characterising this by the action of a "Laplacian", we propose to look at the representation of the centre instead, and develop a version of the Induction Theorem of Savchuk–Schmüdgen that applies in this situation.

For $K = \mathsf{SU}(2)$ and its deformation quantisations, such results are worked out in [63, 160]. A key ingredient there is that the action of the subgroup $\mathbb{T}$ of diagonal matrices in $\mathsf{SU}(2)$ on the universal enveloping algebra of $\mathfrak{k}$ or deformations of it has a commutative fixed-point subalgebra. This circle action allows to define a Fourier decomposition of elements of the enveloping algebra into elements that are homogeneous for the $\mathbb{T}$-action. In this situation, the Induction Theorem produces a C*-hull for the whole *-algebra from one for the (commutative) fixed-point *-subalgebra; this theorem goes back to [160] and is generalised considerably in [113]. An interesting aspect here is that most representations of the fixed-point algebra cannot be induced to a representation of the whole algebra. In many cases, this is where a quantisation happens in the sense that the spectrum of the commutative fixed-point subalgebra is connected, but only a discrete set of characters can be induced. This has to happen for all compact (quantum) groups because we know that they have only countably many irreducible representations. The Induction Theorem in its current form is most useful if a *-algebra admits an action of a torus with commutative fixed-point algebra. But for (quantised) enveloping algebras, this only seems to happen in rank one. To treat higher-rank compact (quantum) groups, we need a generalisation of the Induction Theorem to actions of compact (quantum) groups. The doctoral researcher **Michelle Göbel** of the second cohort is currently working on such a generalisation and further applications of the Induction Theorem for C*-hulls. This should lead to a variant of Nelson's Theorem for compact (quantum) groups that says that a representation of the universal enveloping algebra is integrable if and only if the centre acts by essentially self-adjoint operators. In addition, it is interesting to work out how the inducibility requirement in the induction theorem gives the subset of characters on the centre that correspond to representations of the compact (quantum) group.

While the description through the Induction Theorem that we aim for does not mention a "Laplacian" explicitly, it nevertheless occurs because a representation of the commutative subalgebra is integrable if and only if a strictly positive element of it acts by an essentially self-adjoint operator.

2.7.3 Representations of the string group

The C^*-hull construction may also help to build a representation of the string group that contains the correct analysis to link it to index theory. Index theory is analysed in concrete examples in project area 2.6. The string group is not a compact Lie group, but is best understood as a Lie 2-group. A quick way to define a 2-group is as a tensor category in which all arrows are isomorphisms and all objects have an inverse for the tensor product. The group of objects of the string group is the spin group $\mathsf{Spin}(n)$ covering $\mathsf{SO}(n)$; all its arrows are automorphisms and the automorphism group of each object is a copy of the circle group $\mathbb{T}$. A representation of the string Lie 2-group $\mathsf{String}(G)$ that involves loop group positive energy representations may be used to quantise the classifying space BG equipped with a 2-shifted symplectic structure (see [128]). Such quantisations are then related to the three-dimensional

Chern–Simons theory ([73, 87]).

The 2-shifted symplectic form is described by Segal's 2-form ω on the based loop group ΩG (see [57]). By passing to a central extension of ΩG, it also gives rise to the string 2-group String(G). Thus the latter is interpreted as an analogue of a "prequantisation" for BG in the geometric quantisation scheme. It is expected by physicists that an extended topological quantum field theory for three-dimensional Chern–Simons theory should map the point to the category of positive energy representations of the free loop group [87], equipped with a suitable extra structure.

To perform a geometric quantisation as in [83], we need a higher analogue of a line bundle in the new setting. This should be a bundle of "2-lines" in the world of "2-vector spaces", and these "2-vector spaces" should be what the string 2-group is represented on. Representations of 2-groups take place on objects in a bicategory. Therefore, our geometric quantisation requires a suitable bicategory of "2-vector spaces" and then a line bundle of such objects. Several representations of the string 2-group involving different bicategories of "2-vector spaces" have already been constructed.

Recently, Kristel, Ludewig, and Waldorf have first defined a bicategory of von Neumann algebras, bimodules, and their intertwiners (see [96]) to be their 2-vector spaces, as suggested by Stolz and Teichner in [158]. Then they proceeded in [97] to define a representation of the string 2-group in this bicategory. This representation, however, is not related to positive energy representations of the loop group. Another recent construction in [90] has such a link. It builds on a construction proposed in [121, 156]. It takes place, however, in the purely algebraic bicategory of k-linear categories. Thus the bundle constructed therein does not have a "smooth" or "continuous" structure as geometers and topologists wish.

We propose to add more analysis to the construction by building on recent work by Neeb, Salmasian, and Zellner [123]. They have found a very general method to attach C*-algebras to loop groups and more general infinite-dimensional Lie groups. In particular, they build a "host algebra" for the positive energy representations of a loop group. One of their key results says that the smooth vectors for the "energy operator" are already smooth for the entire infinite-dimensional Lie group. This is again an abstract version of ellipticity.

Since group representations involve only unitaries and thus bounded operators, the host algebras found in [123] are also C*-hulls as defined in [113], which gives them some extra uniqueness and functoriality properties. This should make it possible to build a representation of the string 2-group on the host algebra for the positive energy representations of the loop group. The 2-vector spaces in [96] are a von Neumann algebraic version of the correspondence bicategory of C*-algebras, which has been used as a framework to study various generalisations of group actions on C*-algebras by PIs Meyer and Zhu (see, for instance, [2, 5, 48]).

A group representation in the correspondence bicategory is the same as a Fell bundle over that group. The induction theorem in [113] builds such Fell bundles over a discrete group Γ out of a Γ-graded algebra. This is not yet sufficient because we need to take into account the topology on the string 2-group. Nevertheless, the special case of discrete groups suggests that a string 2-group representation on the host C*-algebra should be related rather directly to suitable operations on the category of positive energy representations.

The development of the different aspects of the above theory offers several attractive questions for doctoral researchers, such as

- Nelson's Theorem for Weinstein groupoids and their Lie algebroids,
- Generalisations and applications of the induction theorem for C*-hulls,

- Representations of the string Lie 2-group via C*-algebras.

2.8 Spectral engineering (Schick, Witt)

In this project, aspects of which are already being pursued by doctoral researcher **Erik Babuschkin** from the second cohort, we are interested in spectral properties of geometric differential operators that are invariant under a cocompact discrete group action. The original example, motivated by solid state physics, is the Laplacian on Euclidean space with a $\mathbb{Z}^n$-invariant potential. The question is: can we achieve a determined band-gap structure of the spectrum of this operator?

Very similar operators occur when introducing and studying analytic L^2-invariants as defined by Atiyah (see 2.3). The starting point is a normal covering $\bar{M} \to M$ of a compact Riemannian manifold M with an action of the group of deck transformations Γ. The relevant operator now is the differential form Laplacian on $\bar{M}$ or, more generally, the lift $\bar{D}$ of an elliptic differential operator D on M. We would like to understand more about the full spectrum of $\bar{D}$. In particular, to what extent can we arrange for a lower bound on the number of gaps (within a spectral range)? For this spectral engineering problem, the role of the potential of the classical problem is now played by the metric (and, in addition, also the topology of M): can we choose the metric so as to achieve a given band-gap structure of the spectrum of $\bar{D}$? Or are there obstructions, forcing the spectrum, say, to be the full line or a half-line? For the scalar Laplacian, this problem goes back at least to [152, Chapter IX, Problem 37].

When the group Γ is $\mathbb{Z}^n$ with $n \neq 0$, Post in [135, 136] solves the problem positively for the scalar Laplacian. Using Fourier analysis in the form of Bloch–Floquet theory, for a given finite energy range Λ, he constructs on arbitrary smooth manifolds M Riemannian metrics g with $\mathbb{Z}^n$-covering $\bar{M}$ and such that the spectrum of the scalar Laplacian on $\bar{M}$ has a prescribed number (and approximate location) of gaps in the interval $[0, \Lambda]$. For a very specific type of manifold, this is refined by Khrabustovskyi, who completely prescribes the band-structure of the spectrum in any finite energy range [95]. Again, this relies heavily on Fourier analysis. This is a periodic analogue of a celebrated result of Colin de Verdière: given n increasing positive numbers, there is a Riemannian metric with precisely these numbers as the first eigenvalues of its scalar Laplacian.

Follow-up work covers more general fundamental groups, culminating in [153] who construct, for an arbitrary covering $\bar{M} \to M$ of a compact manifold and an arbitrary L, a metric on M such that the scalar Laplacian on $L^2(\bar{M})$ for the lifted metric on $\bar{M}$ has at least L gaps in its essential spectrum. Remarkably, there is no condition whatsoever on the covering group.

Despite all this progress, there are quite a number of important open questions. The most intriguing probably is the lack of any example where we can establish an infinite number of bands and gaps in the spectrum if Γ is infinite and therefore the manifold is non-compact. This has been achieved for a very concrete non-compact surface by Lott in [105], but this is an example of finite volume, and not with an infinite cocompact isometry group Γ. The key point is that the essential spectrum is the same as that of a one-dimensional Schrödinger operator, which one can choose conveniently by conformally changing the metric. Lott relies on the classical literature about the spectrum of Schrödinger operators. It is a challenging task to search for a similar explicit construction for cocompact Riemannian metrics.

The scalar Laplacian is only the first in the list of important geometric differential operators. The differential form Laplace–Beltrami operators and the spin Dirac operator of a spin structure offer the next generation of examples. The spectrum of these basic geometric operators should depend strongly on the metric. Only very little is known, however, about spectral engineering for these operators. Recently, Egidi and Post produced metrics on compact manifolds with large gaps in the spectrum of the Hodge Laplacian (a weak analogue for differential forms of Colin de Verdière's result mentioned above), but only on quite special types of manifolds. Metrics with an arbitrary number of gaps in the spectrum of differential form Laplacians and Dirac operators are constructed in [4], but only for $\mathbb{Z}$-symmetry. The construction requires certain topological conditions on a separating hypersurface. It should also be noted here that index theory gives topological obstructions to the existence of gaps in the spectrum of the Dirac operator. The PIs have contributed such obstructions via index theory (for example, in [82, 98, 132]) with a particular emphasis on spectral methods and Fourier decomposition. Therefore, the constructions for general operators need to be more sophisticated than for the scalar Laplacian, where the previous work shows that no such obstructions exist.

Thesis projects will study Dirac and differential form Laplacians with more general symmetry group Γ to identify, on the one hand, obstructions to band-gap structure and, on the other hand, construct examples with many gaps in the spectrum when the obstructions vanish (spectral engineering). The precise results of [95] rely on the full power of Bloch–Floquet theory. In a second line of projects, we will refine these techniques in two directions: to more general symmetry groups Γ (for instance, nilpotent groups as coming up in 2.4 and 2.3) on the one hand, to more general operators (differential form Laplacian, Dirac operator) on the other hand, and construct metrics with prescribed band-gap structure of these operators on Γ-coverings. In all cases, the construction part will involve a family of metrics which degenerates in certain parts of the manifold and such that the spectrum of the operator in question (differential form Laplacian, Dirac operator, ...) converges to the spectrum of a model operator which can be computed explicitly. For abelian groups, Fourier analysis allows to carry out these delicate computations on compact manifolds, which simplifies the situation and therefore will be the first case to be studied. The second cohort doctoral researcher Babuschkin just started a first thesis project on these questions. The presence of obstructions to the existence of gaps is somewhat hard to pin down and will force us to start with special cases, like the n-torus, where we expect that specific constructions like Khrabustovskyi's will allow to control the spectrum of the differential form Laplacians.

A complementary approach to the analysis of spectral properties of Laplacians (and more general operators) is provided by discrete approximations. It is quite subtle to find discretisation techniques that give good approximation results for large parts of the spectrum of the differential operator by the discrete analogues.

Dodziuk and Patodi [61, 62] obtained a refined spectral approximation result for compact manifolds: given any compact Riemannian manifold and finer and finer triangulations which are sufficiently regular, for each k, the kth eigenvalue of the combinatorial Laplacian converges to the kth eigenvalue of the Hodge Laplacian, and this with precise error bounds. In particular, the convergence is uniform on any finite part of the spectrum. This spectral computation uses Rayleigh quotient computations and the precise analysis of the de Rham map and its explicit homotopy inverse constructed by Whitney.

Obviously it does not make sense to aim for a similarly formulated spectral approximation result for the operators on coverings, as they have continuous spectrum in general. A substitute is the spectral density function and, for $\mathbb{Z}^n$-symmetry, the individual terms in the Bloch–Floquet decomposition. Still, one has to formulate the spectral convergence statement carefully. The Rayleigh quotient considerations of the compact case are appropriate for eigenvalue estimates, but again have to be replaced by a more functional analytic treatment for operators with continuous spectrum. We are optimistic that these difficulties can be overcome and propose this as a further thesis topic. One also has to develop the appropriate discrete version of the twisting with a flat representation, which is another subject interesting in its own right.

Some potential doctoral thesis topics in this area are:

- Fine spectral engineering for $\mathbb{Z}^n$-invariant differential form Laplacians using Bloch–Floquet theory, in particular, on $\mathbb{R}^n$, and for differential form Laplacians on coverings with arbitrary deck transformation group,
- Spectral engineering for Dirac operators: index obstructions versus constructions for abelian and non-abelian coverings,
- Spectral approximation via triangulations of manifolds for covering spaces.

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